TAMS32 STOCHASTIC PROCESSES
Komplettering 3

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- Wide sense stationary ARMA processes: existence and representation as linear processes
- Autocorrelation functions for wide sense stationary ARMA processes
- Spectral densities for wide sense stationary ARMA processes
- Appendix: linear homogenous difference equations with constant coefficients

1 Linear processes

Recall that *Kronecker's delta* is defined for all integers $i, j \in \mathbb{Z}$ by:

$$\delta_{i,j} = \begin{cases} 1, & \text{if } i = j; \\ 0, & \text{if } i \neq j. \end{cases} \tag{1.1}$$

A process $\{V_t; t \in \mathbb{Z}\}$ is called **white noise** in discrete time, if it is wide sense stationary with mean 0 and autocorrelation function

$$R_V(\tau) = \sigma^2 \delta_{0,\tau} = \begin{cases} \sigma^2, & \text{if } \tau = 0; \\ 0, & \text{if } \tau \neq 0. \end{cases} \tag{1.2}$$

Definition 1.1 Let $\{V_t; t \in \mathbb{Z}\}$ be white noise in discrete time. Define, for each $t \in \mathbb{Z}$, the random variable

$$Y_t = \sum_{k=0}^{\infty} c_k V_{t-k} = \underset{r\to\infty}{\text{l.i.m.}} \sum_{k=0}^{r} c_k V_{t-k}, \tag{1.3}$$

where $\{c_k; k = 0, 1, \ldots\}$ are real numbers such that $\sum_{k=0}^{\infty} c_k^2 < \infty$ (this condition guarantees that the mean square limit exists, by Theorem 5.1 in Kompletteringshäfte 2). Then, the process $\{Y_t; t \in \mathbb{Z}\}$ is called a **linear process**.

Theorem 1.1 *A linear process* $\{Y_t; t \in \mathbb{Z}\}$ *is wide sense stationary, with mean* $\mu_Y = 0$ *and autocorrelation function*

$$R_Y(\tau) = \sigma^2 \sum_{k=0}^{\infty} c_k c_{k+|\tau|} \qquad \forall \tau \in \mathbb{Z}. \tag{1.4}$$

Proof: By Theorem 4.1 in Kompletteringshäfte 2,

$$E(Y_t) = \sum_{k=0}^{\infty} c_k E(V_{t-k}) = 0 \qquad \forall t \in \mathbb{Z},$$

and, for $\tau = 0, 1, \ldots$,

$$E(Y_t Y_{t+\tau}) = \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} c_k c_l E(V_{t-k} V_{t+\tau-l})$$

$$= \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} c_k c_l \sigma^2 R_V(t + \tau - l - t + k) = \sigma^2 \sum_{k=0}^{\infty} c_k c_{k+\tau} \qquad \forall t \in \mathbb{Z}.$$

For $\tau = -1, -2, \ldots$, we get (using the previous result):

$$E(Y_t Y_{t+\tau}) = E(Y_{t+\tau} Y_{t+\tau+|\tau|}) = \sigma^2 \sum_{k=0}^{\infty} c_k c_{k+|\tau|} \qquad \forall t \in \mathbb{Z}.$$

Since the autocorrelation function does not depend on t, the process is wide sense stationary. ∎

The systematic study of linear processes started with the Ph. D. Thesis of Herman Wold: *A Study in the Analysis of Stationary Time Series*, Almqvist och Wicksell, Uppsala, 1936. We remark that a linear process $\{Y_t; t \in \mathbb{Z}\}$ such that the numbers $\{c_k; k = 0, 1, \ldots\}$ satisfy the stronger condition that $\sum_{k=0}^{\infty} |c_k| < \infty$, is in fact the **output** from a **stable** and **causal LTI** (a linear time-invariant filter), with impulse response $\{c_k; k = 0, 1, \ldots\}$, for which the input is the white noise process $\{V_t; t \in \mathbb{Z}\}$. In this case, the conclusions of Theorem 1.1 follow from Theorem 11.5 in Yates & Goodman.

Since the linear process representation

$$Y_n = \sum_{k=0}^{\infty} c_k V_{n-k} \qquad \forall n \in \mathbb{Z}$$

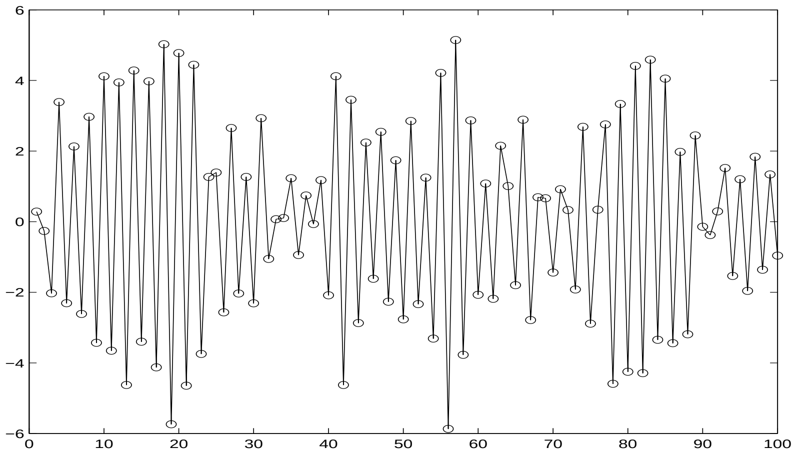
contains infinitely many coefficients, one might think that it would not often in practice be useful for stochastic modelling. However, there are important classes of linear processes which depend only on a finite number of parameters. In the following, we will consider one such class, called the *autoregressive moving average* (ARMA) processes. We begin by studying two special cases: the autoregressive (AR) processes, and the moving average (MA) processes, respectively.

2 Wide sense stationary AR processes

Definition 2.1 A stochastic process $\{Y_n; n \in \mathbb{Z}\}$ is called an **autoregressive process of order** p, or an **AR(p)-process**, if

$$\sum_{i=0}^{p} a_i Y_{n-i} = X_n \qquad \forall n \in \mathbb{Z}, \tag{2.1}$$

where $\{X_n; n \in \mathbb{Z}\}$ is a white noise.



Figur 1: A realisation of an ARMA process.

Equation (2.1) is called an **AR(p) equation**. The polynomial

$$A(s) = \sum_{k=0}^{p} a_k s^k$$

is called an **AR polynomial** (by convention, for an AR polynomial we always have $a_0 = 1$). We first consider the following question: given a white noise process $\{X_n; n \in \mathbb{Z}\}$ and an AR polynomial $A(s)$, when does there exist a process $\{Y_n; n \in \mathbb{Z}\}$ which satisfies equation (2.1)?

Theorem 2.1 *If the polynomial $A(s) = \sum_{k=0}^{p} a_k s^k$ (where $a_0 = 1$) has all its roots outside the complex unit circle, then there exists a unique wide sense stationary process $\{Y_n; n \in \mathbb{Z}\}$ which satisfies the AR(p) equation*

$$\sum_{i=0}^{p} a_i Y_{n-i} = X_n \qquad \forall n \in \mathbb{Z}, \tag{2.2}$$

where $\{X_n; n \in \mathbb{Z}\}$ is a white noise. Moreover, $\{Y_n; n \in \mathbb{Z}\}$ is a linear process:

$$Y_n = \sum_{k=0}^{\infty} c_k X_{n-k} \qquad \forall n \in \mathbb{Z}, \tag{2.3}$$

where $\{c_k; k = 0, 1, \ldots\}$ *are the coefficients of the Maclaurin series expansion of* $C(s) = \frac{1}{A(s)}$*,*

$$\frac{1}{A(s)} = C(s) = \sum_{k=0}^{\infty} c_k s^k \qquad \forall |s| < R_2,$$

and $R_2 > 1$*. In particular,* $\sum_{k=0}^{\infty} |c_k| < \infty$*, so the limit in* (2.3) *exists. The coefficients* $\{c_k; k = 0, 1, \ldots\}$ *also satisfy the homogenous linear difference equation*

$$\sum_{i=0}^{p} a_i c_{k-i} = 0 \qquad \forall k = p, p+1, \ldots$$

and the initial conditions

$$\sum_{i=0}^{k} a_i c_{k-i} = \delta_{0,k} \qquad \forall k = 0, 1, \ldots, p-1,$$

for which the unique solution can be obtained using Theorem 2.3 below.

Proof: We will prove that the process $\{Y_n; n \in \mathbb{Z}\}$, as defined in (2.3), satisfies equation (2.2), but not that it is the unique wide sense stationary process to do so. We will need the following lemma about power series.

Lemma 2.2 *Let* $A(s) = \sum_{k=0}^{\infty} a_k s^k$ *and* $C(s) = \sum_{k=0}^{\infty} b_k s^k$ *be power series which are absolutely convergent for* $|s| < R_1$ *and* $|s| < R_2$*, respectively. Then,* $D(s) = A(s)C(s)$ *has an absolutely convergent Maclaurin series expansion:*

$$D(s) = A(s)C(s) = \sum_{k=0}^{\infty} d_k s^k \qquad \forall |s| < \min(R_1, R_2),$$

where

$$d_k = \sum_{i=0}^{k} a_i c_{k-i} \qquad \forall k = 0, 1, \ldots \tag{2.4}$$

For a proof of Lemma 2.2, see a basic course in analysis. We apply the lemma as follows: let $A(s) = \sum_{k=0}^{p} a_k s^k$ (the AR polynomial), and let

$$C(s) = \frac{1}{A(s)} = \sum_{k=0}^{\infty} c_k s^k \qquad \forall |s| < R_2,$$

where $R_2 > 1$ is the smallest of the moduli of the roots of $A(s)$. (In words, the power series on the right hand side is the Maclaurin series expansion of $C(s) = \frac{1}{A(s)}$, which is absolutely convergent for $|s| < R_2$.) Lemma 2.2 now gives:

$$1 = A(s)C(s) = \sum_{k=0}^{\infty} d_k s^k \qquad \forall |s| < R_2,$$

where

$$d_k = \sum_{i=0}^{k} a_i c_{k-i} \qquad \forall k = 0, 1, \ldots \tag{2.5}$$

The uniqueness theorem for the coefficients of absolutely convergent power series (see a basic course in analysis) now gives us the following infinite system of linear equations for the coefficients $\{c_k; k = 0, 1, \ldots\}$:

$$\begin{aligned} a_0 c_0 &= 1 \\ a_0 c_1 + a_1 c_0 &= 0 \\ a_0 c_2 + a_1 c_1 + a_2 c_0 &= 0 \\ &\vdots \\ a_0 c_n + a_1 c_{n-1} + \ldots + a_p c_{n-p} = \sum_{i=0}^{p} a_i c_{k-i} &= 0 \qquad \forall k = p, p+1, \ldots, \end{aligned} \tag{2.6}$$

where $a_0 = 1$. The last line in (2.6) is a homogenous linear difference equation for $\{c_k; k = 0, 1, \ldots\}$ with constant coefficients, for which the unique solution can be obtained using Theorem 2.3 below.

We now prove that the process $\{Y_t; t \in \mathbb{Z}\}$, defined in (2.3), satisfies equation (2.2). For each $n \in \mathbb{Z}$, it holds that

$$\begin{aligned} \sum_{i=0}^{p} a_i Y_{n-i} &= \\ \underset{r\to\infty}{\text{l.i.m.}} \sum_{k=0}^{r} c_k X_{n-k} + a_1 \underset{r\to\infty}{\text{l.i.m.}} \sum_{k=0}^{r} c_k X_{n-1-k} + \ldots + a_p \underset{r\to\infty}{\text{l.i.m.}} \sum_{k=0}^{r} c_k X_{n-p-k} &= \\ \underset{r\to\infty}{\text{l.i.m.}} (\sum_{k=0}^{r} c_k X_{n-k} + a_1 \sum_{k=0}^{r} c_k X_{n-1-k} + \ldots + a_p \sum_{k=0}^{r} c_k X_{n-p-k}) &= \\ \underset{r\to\infty}{\text{l.i.m.}} \sum_{i=0}^{p} a_i \sum_{k=0}^{r} c_k X_{n-i-k}, & \end{aligned}$$

where, using (2.6), for $r \geq p$,

$$\sum_{i=0}^{p} a_i \sum_{k=0}^{r} c_k X_{n-i-k} = X_n a_0 c_0 + X_{n-1}\left(a_0 c_1 + a_1 c_0\right) + \ldots + X_{n-r} \sum_{i=0}^{p} a_i c_{r-i}$$

$$+ \sum_{i=0}^{p} a_i \sum_{k=r-i+1}^{r} c_k X_{n-i-k} = X_n + 0 + \sum_{i=0}^{p} a_i \sum_{k=r-i+1}^{r} c_k X_{n-i-k}.$$

Next, for $i = 0, \ldots, p$, let $U_i = \sum_{k=r-i+1}^{r} c_k X_{n-i-k}$. It holds that

$$E(U_i) = \sum_{k=r-i+1}^{r} c_k E(X_{n-i-k}) = 0,$$

and

$$V(\sum_{i=0}^{p} a_i U_i) \leq \sum_{i=0}^{p} a_i^2 V(U_i) + 2 \sum_{i=0}^{p-1} \sum_{j=i+1}^{p} |a_i a_j||C(U_i, U_j)|$$

$$\leq \sum_{i=0}^{p} a_i^2 V(U_i) + 2 \sum_{i=0}^{p-1} \sum_{j=i+1}^{p} |a_i a_j| \sqrt{V(U_i) V(U_j)},$$

where we used the Cauchy inequality (see Kompletteringshäfte 2). Also,

$$V(U_i) = \sum_{k=r-i+1}^{r} c_k^2 V(X_{n-i-k}) = \sigma^2 \sum_{k=r-i+1}^{r} c_k^2,$$

which converges to 0 as $r \to \infty$, since $\sum_{k=r-i+1}^{r} c_k^2 < \infty$. These facts together give that

$$\underset{r \to \infty}{\text{l.i.m.}} \sum_{i=0}^{p} a_i \sum_{k=r-i+1}^{r} c_k X_{n-i-k} = 0,$$

so $\{Y_t; t \in \mathbb{Z}\}$ satisfies (2.2). ∎

In order to compute explicitly the coefficients $\{c_k; k = 0, 1, \ldots\}$, the following theorem can be used. (For a proof, see the Appendix.)

Theorem 2.3 *Assume that the real numbers* $\{c_k; k = 0, 1, \ldots\}$ *satisfy the* **homogenous linear difference equation** *with* **constant coefficients***:*

$$\sum_{i=0}^{p} a_i c_{k-i} = 0 \qquad \forall k = p, p+1, \ldots,$$

where $a_1, \ldots, a_p$ *are real numbers, and* $a_p \neq 0$*. It then holds that*

$$c_k = \sum_{\nu=1}^{r} P_\nu(k) u_\nu^k \qquad k = 0, 1, \ldots, \tag{2.7}$$

where $\{u_\nu; \nu = 1, \ldots, r\}$ *are the* **distinct** *roots of the* **characteristic polynomial** $\varphi(s) = a_0 s^p + a_1 s^{p-1} + \ldots + a_p$, $\{m_\nu; \nu = 1, \ldots, r\}$ *are the multiplicities of the distinct roots, and each* $P_\nu(s)$ *is a polynomial of degree* $m_\nu - 1$*. The* $m_1 + m_2 + \ldots + m_\nu = p$ *coefficients of the polynomials* $\{P_\nu(s); \nu = 1, \ldots, r\}$ *can be determined from* p *initial conditions, like those in* (2.6)*.*

Example 2.4 Consider the AR(1) equation

$$Y_n + aY_{n-1} = X_n \qquad \forall n \in \mathbb{Z}, \tag{2.8}$$

where $|a| < 1$. The AR polynomial is $A(s) = 1 + as$, with the only root $s_1 = -\frac{1}{a}$, so by Theorem 2.1, there exists a unique wide sense stationary stochastic process

$$Y_n = \sum_{k=0}^{\infty} c_k X_{n-k} \qquad \forall n \in \mathbb{Z}$$

satisfying (2.8). The coefficients $\{c_k; k = 0, 1, \ldots\}$ satisfy the homogenous linear difference equation with constant coefficients

$$c_n + ac_{n-1} = 0 \qquad \forall n = 1, 2, \ldots,$$

and the initial condition $c_0 = 1$.

We can use Theorem 2.3 to explicitly compute $\{c_k; k = 0, 1, \ldots\}$. The characteristic polynomial is $\varphi(s) = s + a$, with the only root $u_1 = -a$, so

$$c_k = c(-a)^k \qquad k = 0, 1, \ldots \tag{2.9}$$

where $c = P_1(s)$ is a polynomial of degree 0 (a constant). Using the initial condition, we get:

$$c_0 = c(-a)^0 = c = 1.$$

Summing up, we obtain:

$$Y_n = \sum_{k=0}^{\infty} (-a)^k X_{n-k} \qquad \forall n \in \mathbb{Z}. \tag{2.10}$$

It is interesting to note that for the wide sense stationary AR(p) process $\{Y_n; n \in \mathbb{Z}\}$ defined in Theorem 2.1, the autocorrelation function satisfies exactly the same homogenous linear difference equation as the coefficients $\{c_k; k = 0, 1, \ldots\}$, but with a different set of initial conditions.

Theorem 2.5 *Assume that the polynomial* $A(s) = \sum_{k=0}^{p} a_k s^k$ *(where* $a_0 = 1$*) has all its roots outside the complex unit circle, and let* $\{Y_n; n \in \mathbb{Z}\}$ *be the unique wide sense stationary process, defined in* (2.3)*, which satisfies equation* (2.2)*. Then, the autocorrelation function of* $\{Y_n; n \in \mathbb{Z}\}$ *satisfies the homogenous linear difference equation*

$$\sum_{i=0}^{p} a_i R_Y(k-i) = 0 \qquad \forall k = p, p+1, \ldots$$

as well as the initial conditions

$$\sum_{i=0}^{p} a_i R_Y(k-i) = \sigma^2 \delta_{0,k} \qquad \forall k = 0, 1, \ldots, p-1.$$

Proof: Multiplying both sides of equation (2.2) with Y_{n-j}, where $j = 0, 1, \ldots$, gives:

$$\sum_{i=0}^{p} a_i Y_{n-i} Y_{n-j} = Y_{n-j} X_n \qquad \forall n \in \mathbb{Z}.$$

Taking the expected value on both sides gives

$$\sum_{i=0}^{p} a_i R_Y(j-i) = E(Y_{n-j} X_n) \qquad \forall n \in \mathbb{Z}.$$

Using the representation (2.3), and Theorem 4.1 in Kompletteringshäfte 2, we get:

$$E(Y_{n-j} X_n) = E\Big(\sum_{k=0}^{\infty} c_k X_{n-j-k} X_n\Big) = \sum_{k=0}^{\infty} c_k E(X_{n-j-k} X_n)$$

$$= c_0 E(X_{n-j} X_n) = \sigma^2 \delta_{0,j} \qquad \forall n \in \mathbb{Z}.$$

(Recall that $c_0 = 1$.) In particular, R_Y satisfies the homogenous linear difference equation

$$\sum_{i=0}^{p} a_i R_Y(k-i) = 0 \qquad \forall k = p, p+1, \ldots$$

and the initial conditions

$$\sum_{i=0}^{p} a_i R_Y(k-i) = \sigma^2 \delta_{0,k} \qquad \forall k = 0, 1, \ldots, p-1.$$

■

Example 2.6 Consider again the AR(1) equation

$$Y_n + aY_{n-1} = X_n \qquad \forall n \in \mathbb{Z}, \tag{2.11}$$

where $|a| < 1$. It was shown in Example 2.4 that the unique wide sense stationary process which satisfies (2.11) is

$$Y_n = \sum_{k=0}^{\infty} (-a)^k X_{n-k} \qquad \forall n \in \mathbb{Z}.$$

The autocorrelation function of $\{Y_n; n \in \mathbb{Z}\}$ satisfies the same homogenous linear difference equation with constant coefficients as do $\{c_k; k = 0, 1, \ldots\}$, namely:

$$R_Y(k) + aR_Y(k-1) = 0 \qquad \forall k = 1, 2, \ldots,$$

but with a different initial condition:

$$a_0 R_Y(0) + a_1 R_Y(-1) = R_Y(0) + aR_Y(1) = \sigma^2.$$

Using Theorem 2.3, we get

$$R_Y(k) = d(-a)^k \qquad \forall k = 0, 1, \ldots$$

where d is a constant, for which the initial condition yields the equation

$$R_Y(0) + aR_Y(1) = d + ad(-a) = d(1 - a^2) = \sigma^2.$$

Summing up, and using again the fact that $R_Y(-k) = R_Y(k)$, we obtain:

$$R_Y(k) = \frac{\sigma^2(-a)^{|k|}}{1 - a^2} \qquad \forall k \in \mathbb{Z}.$$

We remark that for this example, the autocorrelation function can also be easily computed using Theorem 1.1. (Do this yourself and check that the same answer is obtained!)

3 Moving average (MA) processes

Definition 3.1 A stochastic process $\{Y_n; n \in \mathbb{Z}\}$ is called a **moving average process of order** q, or a **MA(q)-process**, if

$$Y_n = \sum_{i=0}^{q} b_i X_{n-i} \qquad \forall n \in \mathbb{Z}, \tag{3.1}$$

where $\{X_n; n \in \mathbb{Z}\}$ is white noise.

The equation (3.1) is called an **MA equation**. The polynomial

$$B(s) = \sum_{k=0}^{q} b_k s^k$$

is called an **MA polynomial**.

By definition, a moving average process is the output from a FIR (finite impulse response) filter, with impulse response $\{b_k; k = 0, \dots, q\}$, where the input is a white noise process. Therefore, the process $\{Y_n; n \in \mathbb{Z}\}$ is wide sense stationary for any values of the coefficients $\{b_k; k = 0, \dots, q\}$; see Theorem 11.5 in Yates & Goodman. Also, by definition, $\{Y_n; n \in \mathbb{Z}\}$ is a linear process.

Theorem 3.1 *A MA(q) process $\{Y_t; t \in \mathbb{Z}\}$ is wide sense stationary with mean $\mu_Y = 0$, and has the autocorrelation function*

$$R_Y(\tau) = \begin{cases} \sigma^2 \left(b_0 b_{|\tau|} + b_1 b_{|\tau|+1} + \ldots + b_{q-|\tau|} b_q \right), & |\tau| = 0, \dots, q-1; \\ \sigma^2 b_0 b_q, & |\tau| = q; \\ 0, & |\tau| > q. \end{cases} \tag{3.2}$$

Proof: Since the process is linear, by Theorem 1.1 it is wide sense stationary with mean $\mu_Y = 0$, and autocorrelation function

$$R_Y(\tau) = \sigma^2 \sum_{k=0}^{q} b_k b_{k+|\tau|} = \sigma^2 \sum_{k=0}^{q-|\tau|} b_k b_{k+|\tau|} \qquad \forall \tau \in \mathbb{Z}.$$

∎

Comparing the autocorrelation function of a wide sense stationary AR(p) process with that of an MA(q) process, we see that the first function decreases to 0 geometrically as $|\tau| \to \infty$, while the second function has a symmetric "cut-off": it is 0 for $|\tau| > q$.

4 Autoregressive moving average (ARMA) processes

Definition 4.1 A stochastic process $\{Y_n; n \in \mathbb{Z}\}$ is called an **autoregressive moving average process of order** (p, q), or an **ARMA**(p, q) **process**, if

$$\sum_{i=0}^{p} a_i Y_{n-i} = \sum_{i=0}^{q} b_i X_{n-i} \qquad \forall n \in \mathbb{Z} \tag{4.1}$$

where $\{X_n; n \in \mathbb{Z}\}$ is white noise.

Theorem 4.1 *If the polynomial $A(s) = \sum_{k=0}^{p} a_k s^k$ (where $a_0 = 1$) has all its roots outside the complex unit circle, then there exists a unique wide sense stationary process $\{Y_n; n \in \mathbb{Z}\}$ which satisfies equation* (4.1)*. Moreover, $\{Y_n; n \in \mathbb{Z}\}$ is a linear process:*

$$Y_n = \sum_{k=0}^{\infty} c_k X_{n-k} \qquad \forall n \in \mathbb{Z}, \tag{4.2}$$

where $\{c_k; k = 0, 1, \ldots\}$ are the coefficients in the Maclaurin series expansion of $C(s) = \frac{B(s)}{A(s)}$,

$$\frac{B(s)}{A(s)} = C(s) = \sum_{k=0}^{\infty} c_k s^k \qquad \forall |s| < R_2,$$

where $R_2 > 1$. In particular, $\sum_{k=0}^{\infty} |c_k| < \infty$, so the limit in (2.3) *exists. The coefficients $\{c_k; k = 0, 1, \ldots\}$ also satisfy the homogenous linear difference equation*

$$\sum_{i=0}^{p} a_i c_{k-i} = 0 \qquad \forall k \geq \max(p, q+1)$$

and the initial conditions

$$\sum_{i=0}^{k} a_i c_{k-i} = b_k \qquad \forall k = 0, 1, \ldots, \max(p, q+1) - 1,$$

where $b_k = 0$ for $k = q+1, q+2, \ldots$. The unique solution of the difference equation can be obtained using Theorem 2.3.

Proof: Very similar to the proof of Theorem 2.1 Let $A(s) = \sum_{k=0}^{p} a_k s^k$, let $B(s) = \sum_{k=0}^{p} b_k s^k$, and let $C(s) = \frac{B(s)}{A(s)}$, the Maclaurin series expansion of which is absolutely convergent for $|s| < R_2$, where $R_2 > 1$ is the smallest of the moduli of the roots of $A(s)$:

$$C(s) = \frac{B(s)}{A(s)} = \sum_{k=0}^{\infty} c_k s^k \qquad \forall |s| < R_2.$$

Lemma 2.2 now gives:

$$B(s) = \sum_{k=0}^{p} b_k s^k = A(s)C(s) = \sum_{k=0}^{\infty} d_k s^k \qquad \forall |s| < R_2,$$

where

$$d_k = \sum_{i=0}^{k} a_i c_{k-i} \qquad \forall k = 0, 1, \ldots. \tag{4.3}$$

Identifying the coefficients of the power series gives us the following infinite system of linear equations for the coefficients $\{c_k; k = 0, 1, \ldots\}$:

$$\begin{array}{rcll} a_0 c_0 & = & b_0 & \\ a_0 c_1 + a_1 c_0 & = & b_1 & \\ a_0 c_2 + a_1 c_1 + a_2 c_0 & = & b_2 & \\ & \vdots & & \\ \sum_{i=0}^{k} a_i c_{k-i} & = & b_k & k \le \min(p, q) \\ \sum_{i=0}^{p} a_i c_{k-i} & = & b_k & p \le k \le q \\ \sum_{i=0}^{k} a_i c_{k-i} & = & 0 & q + 1 \le k \le p \\ \sum_{i=0}^{p} a_i c_{k-i} & = & 0 & k \ge \max(p, q + 1). \end{array} \tag{4.4}$$

The last line in (4.4) is a homogenous linear difference equation for $\{c_k; k = 0, 1, \ldots\}$ with constant coefficients, for which the unique solution can be obtained using Theorem 2.3. The solution is

$$c_k = \sum_{\nu=1}^{r} P_\nu(k) u_\nu^k, \qquad k \ge \max(0, q + 1 - p). \tag{4.5}$$

where $\{u_\nu; \nu = 1, \ldots, r\}$ are the distinct roots of the characteristic polynomial $\varphi(s) = a_0 s^p + a_1 s^{p-1} + \ldots + a_p$, $\{m_\nu; \nu = 1, \ldots, r\}$ are the multiplicities of the distinct roots, and each $P_\nu(s)$ is a polynomial of degree $m_\nu - 1$. The p coefficients of the polynomials $\{P_\nu(s); \nu = 1, \ldots, r\}$, and the coefficients $\{c_k; k = 0, \ldots, \max(0, q+1-p) - 1\}$, can be determined from the $\max(p, q+1)$ initial conditions given in (4.4). The proof that the process $\{Y_t; t \in \mathbb{Z}\}$, defined in (4.2), satisfies (4.1), is similar to the corresponding proof for AR processes, and is therefore omitted. ∎

As before, for the wide sense stationary ARMA(p, q) process $\{Y_n; n \in \mathbb{Z}\}$ defined in Theorem 4.2, the autocorrelation function satisfies the same homogenous linear difference equation as the coefficients $\{c_k; k = 0, 1, \ldots\}$, but with a different set of initial conditions.

Theorem 4.2 *Assume that the polynomial $A(s) = \sum_{k=0}^{p} a_k s^k$ (where $a_0 = 1$) has all its roots outside the complex unit circle, and let $\{Y_n; n \in \mathbb{Z}\}$ be the unique wide sense stationary process, defined in* (4.2)*, which satisfies equation* (4.1)*. Then, the autocorrelation function of $\{Y_n; n \in \mathbb{Z}\}$ satisfies the homogenous linear difference equation*

$$\sum_{i=0}^{p} a_i R_Y(k-i) = 0 \qquad \forall k \geq \max(p, q+1)$$

and the initial conditions

$$\sum_{i=0}^{p} a_i R_Y(k-i) = \sigma^2 \sum_{i=k}^{q} b_i c_{i-k} \qquad \forall k = 0, \ldots, \max(p, q+1) - 1.$$

Proof: Multiplying both sides of equation (4.1) with Y_{n-j}, where $j = 0, 1, \ldots$, gives:

$$\sum_{i=0}^{p} a_i Y_{n-i} Y_{n-j} = \sum_{i=0}^{q} b_i Y_{n-j} X_{n-i} \qquad \forall n \in \mathbb{Z}.$$

Taking the expected value on both sides gives

$$\sum_{i=0}^{p} a_i R_Y(j-i) = \sum_{i=0}^{q} b_i E(Y_{n-j} X_{n-i}) \qquad \forall n \in \mathbb{Z},$$

where, using the representation (4.2), the right hand side equals

$$\sum_{i=0}^{q} b_i \sum_{l=0}^{\infty} c_l E(X_{n-j} X_{n-i-l}) = \sigma^2 \sum_{i=j}^{q} b_i c_{i-j} \qquad \forall n \in \mathbb{Z}.$$

In particular, R_Y satisfies the homogenous linear difference equation

$$\sum_{i=0}^{p} a_i R_Y(k-i) = 0 \qquad \forall k \geq \max(p, q+1),$$

and the initial conditions

$$\sum_{i=0}^{p} a_i R_Y(k-i) = \sigma^2 \sum_{i=k}^{q} b_i c_{i-k} \qquad \forall k = 0, \ldots, \max(p, q+1) - 1.$$

■

5 The spectral density of a linear process

Recall that for a discrete time wide sense stationary stochastic process $\{Y_n; n \in \mathbb{Z}\}$ such that

$$\sum_{k=1}^{\infty} |R_Y(k)| < \infty, \tag{5.1}$$

the **spectral density** is defined by

$$S_Y(f) = \sum_{k=-\infty}^{\infty} R_Y(k) e^{-j2\pi fk} \qquad \forall -\frac{1}{2} \leq f \leq \frac{1}{2}, \tag{5.2}$$

where $j = \sqrt{-1}$. This expression is also known as the **discrete Fourier transform** of the autocorrelation function. Conversely, given the spectral density, the autocorrelation function is obtained by

$$R_Y(k) = \int_{-1/2}^{1/2} S_Y(f) e^{j2\pi fk} df \qquad \forall k \in \mathbb{Z}.$$

Theorem 5.1 *Let $\{Y_n; n \in \mathbb{Z}\}$ be a linear process*

$$Y_n = \sum_{k=0}^{\infty} c_k V_{n-k} \qquad \forall n \in \mathbb{Z}, \tag{5.3}$$

such that $\sum_{k=0}^{\infty} |c_k| < \infty$, where $\{V_n; n \in \mathbb{Z}\}$ is white noise, and let $C(s) = \sum_{k=0}^{\infty} c_k s^k$. Then,

$$S_Y(f) = \sigma^2 \left| C\left(e^{-j2\pi f}\right)\right|^2 \qquad \forall -\frac{1}{2} \leq f \leq \frac{1}{2}. \tag{5.4}$$

Proof: The white noise process $\{V_n; n \in \mathbb{Z}\}$ has autocorrelation function $R_V(k) = \sigma^2\delta_{0,k}$. Since $\sum_{k=1}^{\infty} |R_V(k)| = \sigma^2 < \infty$, the spectral density is

$$S_V(f) = \sigma^2 \sum_{k=-\infty}^{\infty} \delta_{0,k} e^{-j2\pi fk} = \sigma^2 \qquad \forall -\frac{1}{2} \leq f \leq \frac{1}{2}.$$

In words, white noise has constant spectral density. Since $\{Y_n; n \in \mathbb{Z}\}$ is the output of a stable LTI with impulse response $\{c_k; k = 0, 1, \ldots\}$ for which the input is the white noise process $\{V_n; n \in \mathbb{Z}\}$ (see Section 1), the claim (5.4) follows from Theorem 11.6 in Yates & Goodman. ∎

Example 5.2 An ARMA(p, q) process $\{Y_t; t \in \mathbb{Z}\}$ has spectral density

$$S_Y(f) = \sigma^2 \left| \frac{B\left(e^{-j2\pi f}\right)}{A\left(e^{-j2\pi f}\right)} \right|^2 = \sigma^2 \left| \frac{\sum_{k=0}^{q} b_k e^{-j2\pi fk}}{\sum_{k=0}^{p} a_k e^{-j2\pi fk}} \right|^2 \quad \forall -\frac{1}{2} \leq f \leq \frac{1}{2}. \quad (5.5)$$

This follows from the representation of $\{Y_t; t \in \mathbb{Z}\}$ as a linear process; see Theorem 4.1.

Example 5.3 Consider again the AR(1) equation

$$Y_n + aY_{n-1} = X_n \qquad \forall n \in \mathbb{Z}, \quad (5.6)$$

where $|a| < 1$. It was shown in Example 2.4 that the unique wide sense stationary process which satisfies (5.6) is

$$Y_n = \sum_{k=0}^{\infty} (-a)^k X_{n-k} \qquad \forall n \in \mathbb{Z}.$$

In Example 2.6, it was shown that the autocorrelation function of $\{Y_n; n \in \mathbb{Z}\}$ is

$$R_Y(k) = \frac{\sigma^2(-a)^{|k|}}{1 - a^2} \qquad \forall k \in \mathbb{Z}.$$

By Example 5.2, the spectral density of $\{Y_n; n \in \mathbb{Z}\}$ is

$$S_Y(f) = \frac{\sigma^2}{|1 + ae^{-j2\pi f}|^2} \qquad \forall -\frac{1}{2} \leq f \leq \frac{1}{2}, \quad (5.7)$$

where

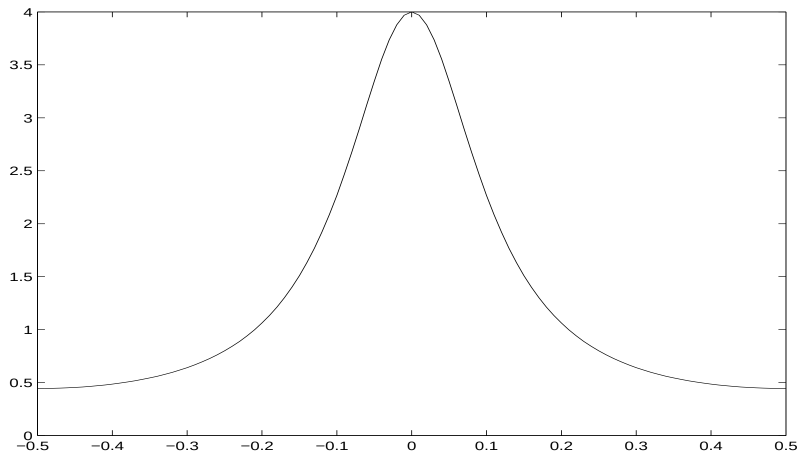
$$\left|1 + ae^{-j2\pi f}\right|^2 = \left(1 + ae^{-j2\pi f}\right)\left(1 + ae^{j2\pi f}\right) = 1 + ae^{j2\pi f} + ae^{-j2\pi f} + a^2$$

$$= 1 + a^2 + a\left(e^{j2\pi f} + e^{-j2\pi f}\right) = 1 + a^2 + 2a\cos(2\pi f) \qquad \forall -\frac{1}{2} \leq f \leq \frac{1}{2},$$

leading to

$$S_Y(f) = \frac{\sigma^2}{1 + a^2 + 2a\cos(2\pi f)} \qquad \forall -\frac{1}{2} \leq f \leq \frac{1}{2}.$$

For an illustration, see Figures 2 and 3.



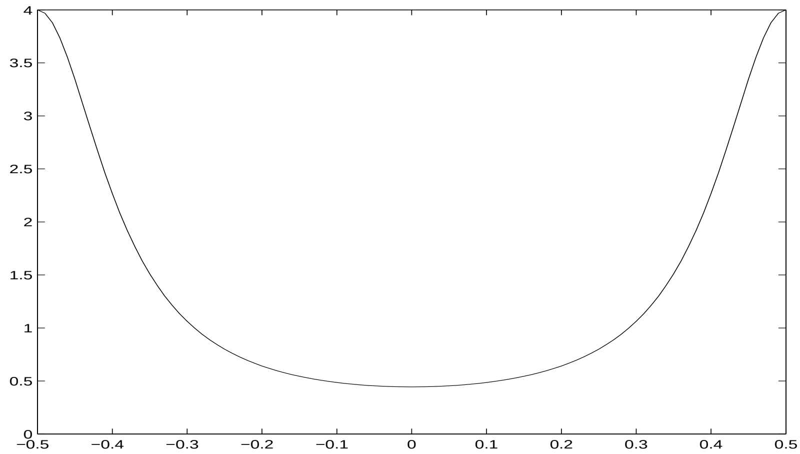
Figur 2: $S_Y(f)$ for an AR(1) process with $a = -0.5$, $\sigma^2 = 1$.

6 Final remarks

One of Herman Wold's most famous results is the **Wold decomposition theorem**, which says that any wide sense stationary process $\{Y_t; t \in \mathbb{Z}\}$ can be written as a sum of a linear process and a so-called **deterministic** wide sense stationary process:

$$Y_n = \sum_{k=0}^{\infty} c_k V_{n-k} + S_n \qquad \forall n \in \mathbb{Z},$$

where $\{V_n; n \in \mathbb{Z}\}$ is a white noise process, and $\sum_{k=0}^{\infty} c_k^2 < \infty$. The process $\{S_n; n \in \mathbb{Z}\}$ is deterministic in the sense that for each $n \in \mathbb{Z}$, S_n is the mean square limit of a sequence of linear combinations of random variables in the



Figur 3: $S_Y(f)$ for an AR(1) process with $a = 0.5$, $\sigma^2 = 1$.

(infinite) set $\{S_m; m \leq n-1\}$. So, if $\{S_m; m \leq n-1\}$ is known, then S_n is also "known" in the sense that it can be approximated arbitrarily well (in mean square) by a linear combination of elements in $\{S_m; m \leq n-1\}$.

The theory of linear processes, and of ARMA processes in particular, has seen a rapid development since the appearance of Wold's Thesis. They have been used to model an ever increasing number of phenomena within the areas of science, technology and economics. Many generalizations and variations on ARMA processes have been suggested, for example multivariate ARMA processes, ARIMA processes, RCA (random coefficient autoregressive) processes, and ARCH-and GARCH (generalized autoregressive conditionally heteroscedastic) processes, which have found use within the area of mathematical finance. For more information, see:

- T.W. Anderson: *Time Series Analysis.* John Wiley and Sons, Inc. New York 1971.
- G.E.P. Box & G.M. Jenkins: *Time Series Analysis.* Holden-Day, San Francisco, 1970.
- P.J. Brockwell & R.A. Davis: *Introduction to Time Series and Forecasting* Springer-Verlag, New York, 2002.
- U. Hjorth: *Stochastic processes. Korrelations- och spektralteori.* Studentlitteratur, Lund 1987.

- K.J. Åström: *Introduction to Stochastic Control Theory*. Academic Press, New York, 1970.

A Appendix: Linear difference equations with constant coefficients

Let $y = \{y_n; n = 0, 1, \ldots\}$ be a real valued sequence (= a sequence of real numbers). Denote by $\triangle$ the **(forward) difference operator**, which is a mapping from the real valued sequences to the real valued sequences, defined by

$$\triangle y = \{(\triangle y)_n; n = 0, 1, \ldots\} = \{y_{n+1} - y_n; n = 0, 1, \ldots\} \tag{A.1}$$

Denote by $\triangle^k$ the kth power of $\triangle$, defined (recursively) by

$$\triangle^k y = \triangle(\triangle^{k-1} y) \qquad \forall k = 1, 2, 3, \ldots . \tag{A.2}$$

By induction, it can be shown that

$$\triangle^k y = \left\{ y_{n+k} - \binom{k}{1} y_{n+k-1} + \cdots + (-1)^k y_n; n = 0, 1, \ldots \right\} \tag{A.3}$$

Definition A.1 An **(ordinary) difference equation** is an equation of the following type:

$$G(n, y_n, (\triangle y)_n, \ldots, (\triangle^k y)_n) = 0 \qquad \forall n = 0, 1, \ldots, \tag{A.4}$$

where $G : \mathbb{R}^{k+2} \to \mathbb{R}$ a function. Equivalently, using (A.3), an ordinary difference equation can be written in the following form:

$$F(n, y_n, y_{n+1}, \ldots, y_{n+k}) = 0 \qquad \forall n = 0, 1, \ldots, \tag{A.5}$$

where $F : \mathbb{R}^{k+2} \to \mathbb{R}$ a function.

Definition A.2 If F in (A.5) takes the form

$$f_{0,n} y_n + f_{1,n} y_{n+1} + \cdots + f_{k,n} y_{n+k} = g_n \qquad \forall n = 0, 1, \ldots, \tag{A.6}$$

where $f_i = \{f_{i,n}; n = 0, 1, \ldots\}$, $i = 0, 1, \ldots, k$, and $g = \{g_n; n = 0, 1, \ldots\}$ are real valued sequences, and neither f_0 nor f_k are identically 0, the equation is called a **linear (ordinary) difference equation of order** k. For brevity, we call an equation such as (A.6) a linear difference equation of order k.

Definition A.3 If g in (A.6) is identically 0, the linear difference equation is called **homogenous**, otherwise **inhomogenous**.

Definition A.4 If f_i is equal to a constant a_i for $i = 0, 1, \ldots k$ (that is, if $f_{i,n} = a_i$ for $n = 0, 1, \ldots$ and $i = 0, 1, \ldots k$), the equation is called a **linear difference equation of order** k with **constant coefficients**. In this case, and if the equation is homogenous, it can be written in the form

$$a_0 y_n + a_1 y_{n+1} + \cdots + a_k y_{n+k} = 0 \qquad \forall n = 0, 1, 2, \ldots \tag{A.7}$$

Consider now a real valued sequence y of the particular form

$$y_n = u^n \qquad \forall n = 0, 1, 2, \ldots,$$

where $u \in \mathbb{R}$. This sequence is a solution to (A.7) if and only if

$$a_0 u^n + a_1 u^{n+1} + \cdots + a_k u^{n+k} = u^n (a_0 + a_1 u^1 + \cdots + a_k u^k) = 0 \quad \forall n = 0, 1, 2, \ldots,$$

that is, if and only if u is a root of the polynomial

$$\varphi(u) = a_0 + a_1 u + \cdots + a_k u^k,$$

which is known as the **characteristic polynomial** of the difference equation (A.7). The equation $\varphi(u) = 0$ is called the **characteristic equation** of (A.7). We have shown the following lemma.

Lemma A.1 *If u_1 is a root of the characteristic polynomial $\varphi(u) = a_0 + a_1 u + \cdots + a_k u^k$, the real valued sequence*

$$y_n = c u_1^n \qquad \forall n = 0, 1, 2, \ldots, \tag{A.8}$$

where c is a real constant, is a solution to the difference equation (A.7).

For the linear difference equation (A.7), it is easily seen that any linear combination of two solutions to the equation is also a solution. This proves the following theorem.

Theorem A.2 If the characteristic polynomial $\varphi(u) = a_0 + a_1 u + \cdots + a_k u^k$ has k **distinct** roots $u_1, u_2, \ldots, u_k$, then the real valued sequence

$$y_n = c_1 u_1^n + c_2 u_2^n + \cdots + c_k u_k^n \qquad \forall n = 0, 1, 2, \ldots \tag{A.9}$$

where $c_1, c_2, \ldots, c_k$ are real constants, is a solution to the difference equation (A.7).

Example A.3 Consider the linear homogenous difference equation with constant coefficients

$$6y_n - 11y_{n+1} + 6y_{n+2} - y_{n+3} = 0 \qquad \forall n = 0, 1, 2, \dots \tag{A.10}$$

The corresponding characteristic polynomial is

$$\varphi(u) = 6 - 11u + 6u^2 - u^3.$$

This polynomial can be shown to have the roots $u_1 = 1$, $u_2 = 2$ and $u_3 = 3$. By Theorem A.2, any sequence of the type

$$y_n = c_1 + c_2 2^n + c_k 3^n \qquad \forall n = 0, 1, 2, \dots, \tag{A.11}$$

where c_1, c_2 and c_3 are real constants, is a solution to the difference equation (A.7). Assume now that we require that the solution to (A.7) should satisfy the following **initial conditions**:

$$y_0 = 3, \qquad y_1 = 6, \qquad y_2 = 14.$$

Then, by (A.11), the constants c_1, c_2 and c_3 must satisfy the following system of linear equations:

$$\begin{aligned} c_1 + c_2 + c_3 &= y_0 = 3 \\ c_1 + 2c_2 + 3c_3 &= y_1 = 6 \\ c_1 + 4c_2 + 9c_3 &= y_2 = 14 \end{aligned}$$

This system has the unique solution

$$c_1 = c_2 = c_3 = 1.$$

Hence, the only solution to (A.7) of the type (A.11) satisfying the given initial conditions is

$$y_n = 1 + 2^n + 3^n \qquad \forall n = 0, 1, 2, \dots.$$

Lemma A.4 *If u_1 is a* **root with multiplicity 2** *of the characteristic polynomial $\varphi(u) = a_0 + a_1 u + \cdots + a_k u^k$, the real valued sequence*

$$y_n = cnu_1^n \qquad \forall n = 0, 1, 2, \dots, \tag{A.12}$$

where c is a real constant, is a solution to the difference equation (A.7).

Proof: It is clear that

$$a_0 y_n + a_1 y_{n+1} + a_2 y_{n+2} + \cdots + a_k y_{n+k}$$
$$= c a_0 n u_1^n + c a_1 (n+1) u_1^{n+1} + \cdots + c a_k (n+k) u_1^{n+k}$$
$$= c u_1 \big(a_0 n u_1^{n-1} + a_1 (n+1) u_1^n + \cdots + a_k (n+k) u_1^{n+k-1} \big) = c u_1 Q_n'(u_1),$$

where $Q_n(u) = \varphi(u) u^n$. Since u_1 has multiplicity 2 as a root of the characteristic polynomial, we can write

$$\varphi(u) = (u - u_1)^2 \beta(u),$$

where $\beta(u)$ is a polynomial such that $\beta(u_1) \neq 0$. This gives:

$$\begin{aligned} Q_n'(u) &= \big((u-u_1) 2\beta(u) + (u-u_1)^2 \beta'(u)\big) u^n + (u-u_1)^2 \beta(u) n u^{n-1} \\ &= (u-u_1) h_n(u), \end{aligned}$$

where $h_n(u)$ is another polynomial such that $h_n(u_1) = 2\beta(u_1) \neq 0$. Therefore, u_1 is a root of the polynomial $Q_n'(u)$ (with multiplicity 1). ∎

Lemma A.5 *If u_1 is a root with multiplicity 2 of the characteristic polynomial $\varphi(u) = a_0 + a_1 u + \cdots + a_k u^k$, the real valued sequence*

$$y_n = (cn + d) u_1^n \qquad \forall n = 0, 1, 2, \ldots, \tag{A.13}$$

where c and d are real constants, is a solution to the difference equation (A.7).

Proof: This follows from Lemma A.1, Lemma A.4, and the fact that any linear combination of two solutions to the difference equation (A.7) is also a solution. ∎

Example A.6 Consider the linear homogenous difference equation with constant coefficients

$$4y_n + 4y_{n+1} - 3y_{n+2} - 2y_{n+3} + y_{n+4} = 0 \qquad n = 0, 1, 2, \ldots \tag{A.14}$$

The corresponding characteristic polynomial is

$$\varphi(u) = 4 + 4u - 3u^2 - 2u^3 + u^4 = (u+1)^2 (u-2)^2.$$

By Lemma A.5, any sequence of the type

$$y_n = (A_1 n + B_1)(-1)^n + (A_2 n + B_2)2^n \qquad \forall n = 0, 1, 2, \ldots,$$

where A_1, A_2, B_1 and B_2 are real constants, is a solution to the difference equation (A.7). The four constants can be determined if we require that the solution satisfies four initial conditions.

Lemma A.7 If u_1 is a **root with multiplicity** m of the characteristic polynomial $\varphi(u) = a_0 + a_1 u + \cdots + a_k u^k$, the real valued sequence

$$y_n = P_m(n) u_1^n \qquad \forall n = 0, 1, 2, \ldots, \tag{A.15}$$

where $P_m(\cdot)$ is a real valued polynomial of degree $m-1$, is a solution to the difference equation (A.7).

Proof: Just as in the proof of Lemma A.4, we define the auxiliary function $Q_n(u) = \varphi(u) u^n$. Since u_1 has multiplicity m as a root of the characteristic polynomial,

$$\varphi(u) = (u - u_1)^m \beta(u),$$

where $\beta(u)$ is a polynomial such that $\beta(u_1) \neq 0$. The same argument as in the proof of Lemma A.4 shows that

$$Q(u_1) = Q'(u_1) = \cdots = Q^{m-1}(u_1) = 0. \tag{A.16}$$

Next, we express the polynomial $P_m(s)$ in the form

$$P_m(s) = b_0 + b_1 s + b_2 s(s-1) + \cdots + b_{m-1} s(s-1)(s-2)\ldots(s-m+2),$$

where $b_0, b_1, \ldots, b_n$ are real numbers. Clearly, any polynomial of degree $m-1$ can be uniquely expressed in this way. Note also that

$$n(n-1)\cdots(n-j+1) = 0 \qquad \forall n = 0, 1, \ldots, j-1,$$

for all $j = 1, 2, \ldots$. We can now write:

$$y_n = P_m(n) u_1^n = \sum_{j=0}^{m-1} b_j \big(n(n-1)\cdots(n-j+1)\big) u_1^n$$

$$= \sum_{j=0}^{m-1} b_j u_1^j \big[\frac{\mathrm{d}^j}{\mathrm{d}u^j} u^n\big]_{u=u_1} \qquad \forall n = 0, 1, 2, \ldots$$

(please verify the last equality!) This finally leads to:

$$\begin{aligned}
\sum_{i=0}^{k} a_i y_{n+i} &= \sum_{i=0}^{k} a_i \sum_{j=0}^{m-1} u_1^j b_j \big[\frac{\mathrm{d}^j}{\mathrm{d}u^j} u^{n+i}\big]_{u=u_1} = \\
&= \sum_{j=0}^{m-1} u_1^j b_j \big[\frac{\mathrm{d}^j}{\mathrm{d}u^j} \sum_{i=0}^{k} a_i u^{n+i}\big]_{u=u_1} = \\
&= \sum_{j=0}^{m-1} u_1^j b_j \big[\frac{\mathrm{d}^j}{\mathrm{d}u^j} Q_n(u)\big]_{u=u_1} = 0 \qquad \text{(A.17)}
\end{aligned}$$

by (A.16). ∎

We are now ready to state our main result concerning the general solution of a linear homogenous difference equation of order k with constant coefficients.

Theorem A.8 Let $\varphi(u) = a_0 + a_1 u + \cdots + a_k u^k$ be the characteristic polynomial of the linear honogenous difference equation (A.7). If $\varphi(u)$ has r **distinct** roots $u_1, u_2, \ldots, u_r$, with multiplicities $\{m_1, m_2, \ldots, m_r\}$, then the real valued sequence

$$y_n = \sum_{\nu=1}^{r} P_\nu(n) u_\nu^n \qquad \forall n = 0, 1, 2, \ldots \qquad \text{(A.18)}$$

where, for each $\nu = 1, \ldots, r$, $P_\nu(s)$ is a polynomial of degree $m_\nu - 1$, is a solution to the difference equation (A.7). Moreover, any solution to (A.7) can be written in the form (A.18).

Proof: The first claim follows from Lemma A.7 and the fact that any linear combination of two solutions to the difference equation (A.7) is also a solution. The proof of the second claim, that any solution can be written in this form, is omitted. ∎

Example A.9 Consider the linear homogenous difference equation with constant coefficients

$$y_n + 3y_{n+1} + 3y_{n+2} + y_{n+3} = 0 \qquad \forall n = 0, 1, 2, \ldots$$

The corresponding characteristic polynomial is

$$\varphi(u) = 1 + 3u + 3u^2 + u^3 = (u+1)^3$$

By Theorem A.8, any sequence of the type

$$y_n = (A_2 n^2 + A_1 n + A_0)(-1)^n \qquad \forall n = 0, 1, 2, \ldots, \tag{A.19}$$

where A_2, A_1, and A_0 are real constants, is a solution to the difference equation. Let us require that the solution satisfies three initial conditions: $y_0 = 1$, $y_1 = 2$, $y_2 = -9$. Then, the constants A_2, A_1, and A_0 must satisfy the following linear system of equations:

$$\begin{aligned} A_0(-1)^0 &= y_0 = 1 \\ (A_2 1^2 + A_1 1 + A_0)(-1)^1 &= y_1 = 2 \\ (A_2 2^2 + A_1 2 + A_0)(-1)^2 &= y_2 = -9 \end{aligned}$$

The system has the unique solution $A_0 = 1$, $A_1 = -1$, $A_2 = -2$. Hence, the only solution to the difference equation of the type (A.19) satisfying the given initial conditions is

$$y_n = (-2n^2 - n + 1)(-1)^n \qquad \forall n = 0, 1, 2, \ldots.$$

We remark that a characteristic polynomial $\varphi(u)$ can have complex roots. However, since the (constant) coefficients of the linear difference equation (A.7) are real numbers, nonreal roots always occur in conjugate pairs. Furthermore, since also the initial values are real numbers, it can be shown by induction that the solution $\{y_t; t = 0, 1, \ldots\}$ to the difference equation must be a real valued sequence.

Example A.10 Consider the linear homogenous difference equation with constant coefficients

$$y_n + y_{n+2} = 0 \qquad \forall n = 0, 1, 2, \ldots$$

The corresponding characteristic polynomial $\varphi(u) = u^2 + 1$ has two complex roots, which are complex conjugates:

$$u_1 = i = e^{i\pi/2} \qquad u_2 = -i = e^{-i\pi/2}$$

By Theorem A.8, any sequence of the type

$$y_n = c_1 e^{in\pi/2} + c_2 e^{-in\pi/2} \qquad \forall n = 0, 1, 2, \ldots, \tag{A.20}$$

where c_1 and c_2 are (possibly complex) constants, is a solution to the difference equation. Using Euler's formulæ, this can be written as

$$y_n = (c_1 + c_2)\cos(n\pi/2) + i(c_1 - c_2)\sin(n\pi/2) \qquad \forall n = 0, 1, 2, \ldots$$

Since $\{y_n; n = 0, 1, \ldots\}$ is a real valued sequence, we see (by choosing $n = 0$ and $n = 1$) that c_1 and c_2 must be complex conjugates. We can therefore write:

$$y_n = a\cos(n\pi/2) + b\sin(n\pi/2) \qquad \forall n = 0, 1, 2, \ldots,$$

where $a = 2\mathrm{Re}(c_1) = 2\mathrm{Re}(c_2)$ and $b = -2\mathrm{Im}(c_1) = 2\mathrm{Im}(c_2)$. Assume that the solution should satisfy the initial conditions $y_0 = 1$ och $y_1 = 0$. Then, we see that $a = 1$ och $b = 0$, so the only solution to the difference equation of the type (A.20) satisfying the given initial conditions is

$$y_n = \cos(n\pi/2) \qquad \forall n = 0, 1, 2, \ldots.$$

The fact that this is a solution can also be seen as follows:

$$y_{n+2} = \cos((n+2)\pi/2) = \cos(n\pi/2 + \pi) = -\cos(n\pi/2) \qquad \forall n = 0, 1, 2, \ldots$$