TAMS46: Probability Theory (Second Course) | Provkod: TEN1 | 30 October 2020, 08:00-12:00

Examiner: Xiangfeng Yang (013-285788). **Things allowed**: a calculator, an English-Swedish dictionary.
Scores rating (Betygsgränser): 8-11 points giving rate 3; 11.5-14.5 points giving rate 4; 15-18 points giving rate 5.

1 (3 points)

Let X and Y be independent $Exp(1)$-distributed random variables. Show that $X/(X+Y)$ and $X+Y$ are independent, and find their density functions $f_{X/(X+Y)}(u)$ and $f_{X+Y}(v)$.

2 (3 points)

Let Y be a Binomial random variable with a random parameter X as follows:

$$Y \mid X = x \sim Bin(n, x), \quad \text{with } X \sim U(0,1). \quad \Big[\text{This can be also written as } Y \mid X \sim Bin(n, X)\Big]$$

Compute the expectation $E(Y)$ of Y and the covariance $Cov(X,Y)$ of X and Y.

3 (3 points)

Prove, with the aid of suitable transforms, that if $X \sim Bin(n,p)$ and $Y \sim Bin(m,p)$ are independent, then $X + Y \sim Bin(n+m, p)$.

4 (3 points)

The random variables $X_1, X_2, \ldots, X_n, Y_1, Y_2, \ldots, Y_n$ are independent $U(0,1)$-distributed.
(4.1) Find the density function $f_{X_{(n)}}(x)$ of $X_{(n)}$, where $X_{(n)} = \max\{X_1, X_2, \ldots, X_n\}$.
(4.2) Find the probability $P\left(\frac{\max\{X_{(n)}, Y_{(n)}\}}{\min\{X_{(n)}, Y_{(n)}\}} \geq 2\right)$.

5 (3 points)

Let $\mathbf{X} = \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} \sim N(\mathbf{0}, \mathbf{\Lambda})$ with $\mathbf{\Lambda} = \begin{pmatrix} 3 & 1 \\ 1 & 2 \end{pmatrix}$, that is, the density function is $f_{\mathbf{X}}(\mathbf{x}) = \frac{1}{2\pi \cdot \sqrt{det(\mathbf{\Lambda})}} \exp\left\{-\frac{1}{2}\mathbf{x}'\mathbf{\Lambda}^{-1}\mathbf{x}\right\}$.
Find the conditional density function $f_{X_1+X_2 \mid X_1 - X_2 = 0}(x)$ of $X_1 + X_2$ given that $X_1 - X_2 = 0$.
(Hints: you might need to use the inverse formula $\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$.)

6 (3 points)

(6.1) Let $P(Z_n = 0) = 1 - \frac{1}{n}, P(Z_n = 1) = \frac{1}{2n}$ and $P(Z_n = -1) = \frac{1}{2n}$ for $n \geq 1$. Prove that $Z_n \xrightarrow{p} 0$.
(6.2) Let Y be a Normal random variable with a random parameter X as follows:

$$Y \mid X = x \sim N(0, x), \quad \text{with } X \sim Po(\lambda). \quad \Big[\text{This can be also written as } Y \mid X \sim N(0, X)\Big]$$

Prove that $Y/\sqrt{\lambda} \xrightarrow{d} N(0,1)$ as $\lambda \to \infty$.
(Hint: use the characteristic function of $Y/\sqrt{\lambda}$, and a fact $e^x = 1 + x + o(x)$ as $x \to 0$.)

Discrete Distributions

Followingis a list of discrete distributions, abbreviations, their probability functions, means, variances, and characteristic functions. An asterisk (*) indicates that the expression is too complicated to present here; in some cases a closed formula does not even exist.

Distribution, notation	Probability function	$E\,X$	$\mathrm{Var}\,X$	$\varphi_X(t)$
One point $\delta(a)$	$p(a)=1$	a	0	e^{ita}
Symmetric Bernoulli	$p(-1)=p(1)=\frac{1}{2}$	0	1	$\cos t$
Bernoulli $\mathrm{Be}(p)$, $0\le p\le 1$	$p(0)=q,\ p(1)=p;\ q=1-p$	p	pq	$q+pe^{it}$
Binomial $\mathrm{Bin}(n,p)$, $n=1,2,\ldots$, $0\le p\le 1$	$p(k)=\binom{n}{k}p^kq^{n-k},\ k=0,1,\ldots,n;\ q=1-p$	np	npq	$(q+pe^{it})^n$
Geometric $\mathrm{Ge}(p)$, $0\le p\le 1$	$p(k)=pq^k,\ k=0,1,2,\ldots;\ q=1-p$	$\frac{q}{p}$	$\frac{q}{p^2}$	$\frac{p}{1-qe^{it}}$
First success $\mathrm{Fs}(p)$, $0\le p\le 1$	$p(k)=pq^{k-1},\ k=1,2,\ldots;\ q=1-p$	$\frac{1}{p}$	$\frac{q}{p^2}$	$\frac{pe^{it}}{1-qe^{it}}$
Negative binomial $\mathrm{NBin}(n,p)$, $n=1,2,3,\ldots$, $0\le p\le 1$	$p(k)=\binom{n+k-1}{k}p^nq^k,\ k=0,1,2,\ldots;$ $q=1-p$	$n\frac{q}{p}$	$n\frac{q}{p^2}$	$(\frac{p}{1-qe^{it}})^n$
Poisson $\mathrm{Po}(m)$, $m>0$	$p(k)=e^{-m}\frac{m^k}{k!},\ k=0,1,2,\ldots$	m	m	$e^{m(e^{it}-1)}$
Hypergeometric $H(N,n,p)$, $n=0,1,\ldots,N$, $N=1,2,\ldots$, $p=0,\frac{1}{N},\frac{2}{N},\ldots,1$	$p(k)=\frac{\binom{Np}{k}\binom{Nq}{n-k}}{\binom{N}{n}},\quad k=0,1,\ldots,Np;$ $q=1-p;$ $n-k=0,\ldots,Nq$	np	$npq\,\frac{N-n}{N-1}$	*

Continuous Distributions

Following is a list of some continuous distributions, abbreviations, their densities, means, variances, and characteristic functions. An asterisk (*) indicates that the expression is too complicated to present here; in some cases a closed formula does not even exist.

Distribution, notation	Density	$E\,X$	$\mathrm{Var}\,X$	$\varphi_X(t)$
Uniform/Rectangular				
$U(a,b)$	$f(x)=\frac{1}{b-a},\ a<x<b$	$\frac{1}{2}(a+b)$	$\frac{1}{12}(b-a)^2$	$\frac{e^{itb}-e^{ita}}{it(b-a)}$
$U(0,1)$	$f(x)=1,\ 0<x<1$	$\frac{1}{2}$	$\frac{1}{12}$	$\frac{e^{it}-1}{it}$
$U(-1,1)$	$f(x)=\frac{1}{2},\ \lvert x\rvert<1$	0	$\frac{1}{3}$	$\frac{\sin t}{t}$
Triangular				
$\mathrm{Tri}(a,b)$	$f(x)=\frac{2}{b-a}\left(1-\frac{2}{b-a}\left\lvert x-\frac{a+b}{2}\right\rvert\right)$, $a<x<b$	$\frac{1}{2}(a+b)$	$\frac{1}{24}(b-a)^2$	$\left(\frac{e^{itb/2}-e^{ita/2}}{\frac{1}{2}it(b-a)}\right)^2$
$\mathrm{Tri}(-1,1)$	$f(x)=1-\lvert x\rvert,\ \lvert x\rvert<1$	0	$\frac{1}{6}$	$\left(\frac{\sin\frac{t}{2}}{\frac{t}{2}}\right)^2$
Exponential $\mathrm{Exp}(a)$, $a>0$	$f(x)=\frac{1}{a}e^{-x/a},\ x>0$	a	a^2	$\frac{1}{1-ait}$
Gamma $\Gamma(p,a)$, $a>0$, $p>0$	$f(x)=\frac{1}{\Gamma(p)}x^{p-1}\frac{1}{a^p}e^{-x/a},\ x>0$	pa	pa^2	$\frac{1}{(1-ait)^p}$
Chi-square $\chi^2(n)$, $n=1,2,3,\ldots$	$f(x)=\frac{1}{\Gamma(\frac{n}{2})}x^{\frac{1}{2}n-1}\left(\frac{1}{2}\right)^{n/2}e^{-x/2},\ x>0$	n	$2n$	$\frac{1}{(1-2it)^{n/2}}$
Laplace $L(a)$, $a>0$	$f(x)=\frac{1}{2a}e^{-\lvert x\rvert/a},\ -\infty<x<\infty$	0	$2a^2$	$\frac{1}{1+a^2t^2}$
Beta $\beta(r,s)$, $r,s>0$	$f(x)=\frac{\Gamma(r+s)}{\Gamma(r)\Gamma(s)}x^{r-1}(1-x)^{s-1}$, $0<x<1$	$\frac{r}{r+s}$	$\frac{rs}{(r+s)^2(r+s+1)}$	*

Continuous Distributions (continued)

Distribution, notation	Density	$E\,X$	$\operatorname{Var} X$	$\varphi_X(t)$
Weibull $W(\alpha,\beta)$, $\alpha,\beta>0$	$f(x)=\frac{1}{\alpha\beta}x^{(1/\beta)-1}\,e^{-x^{1/\beta}/\alpha},\ x>0$	$\alpha^\beta\,\Gamma(\beta+1)$	$a^{2\beta}\big(\Gamma(2\beta+1)-\Gamma(\beta+1)^2\big)$	*
Rayleigh $\mathrm{Ra}\,(\alpha)$, $\alpha>0$	$f(x)=\frac{2}{\alpha}\,x\,e^{-x^2/\alpha},\ x>0$	$\frac{1}{2}\sqrt{\pi\alpha}$	$\alpha(1-\frac{1}{4}\pi)$	*
Normal $N(\mu,\sigma^2)$, $-\infty<\mu<\infty$, $\sigma>0$	$f(x)=\frac{1}{\sigma\sqrt{2\pi}}\,e^{-\frac{1}{2}(x-\mu)^2/\sigma^2},\ -\infty<x<\infty$	μ	σ^2	$e^{i\mu t-\frac{1}{2}t^2\sigma^2}$
$N(0,1)$	$f(x)=\dfrac{1}{\sqrt{2\pi}}\,e^{-x^2/2},\ -\infty<x<\infty$	0	1	$e^{-t^2/2}$
Log-normal $LN(\mu,\sigma^2)$, $-\infty<\mu<\infty,\ \sigma>0$	$f(x)=\dfrac{1}{\sigma x\sqrt{2\pi}}\,e^{-\frac{1}{2}(\log x-\mu)^2/\sigma^2},\ x>0$	$e^{\mu+\frac{1}{2}\sigma^2}$	$e^{2\mu}\big(e^{2\sigma^2}-e^{\sigma^2}\big)$	*
(Student's) t $t(n)$, $n=1,2,\ldots$	$f(x)=\frac{\Gamma(\frac{n+1}{2})}{\sqrt{\pi n}\Gamma(\frac{n}{2})}\cdot d\frac{1}{(1+\frac{x^2}{n})^{(n+1)/2}},\ -\infty<x<\infty$	0	$\dfrac{n}{n-2},\ n>2$	*
(Fisher's) F $F(m,n)$, $m,n=1,2,\ldots$	$f(x)=\frac{\Gamma(\frac{m+n}{2})(\frac{m}{n})^{m/2}}{\Gamma(\frac{m}{2})\Gamma(\frac{n}{2})}\cdot\frac{x^{m/2-1}}{(1+\frac{mx}{n})^{(m+n)/2}},\ x>0$	$\frac{n}{n-2},\ n>2$	$\frac{n^2(m+2)}{m(n-2)(n-4)}-\left(\frac{n}{n-2}\right)^2,\ n>4$	*

Continuous Distributions (continued)

Distribution, notation	Density	$E\,X$	$\mathrm{Var}\,X$	$\varphi_X(t)$
Cauchy				
$C(m,a)$	$f(x) = \frac{1}{\pi} \cdot \frac{a}{a^2+(x-m)^2},\ -\infty < x < \infty$	$\nexists$	$\nexists$	$e^{imt-a\lvert t\rvert}$
$C(0,1)$	$f(x) = \frac{1}{\pi} \cdot \frac{1}{1+x^2},\ -\infty < x < \infty$	$\nexists$	$\nexists$	$e^{-\lvert t\rvert}$
Pareto	$f(x) = \frac{\alpha k^\alpha}{x^{\alpha+1}},\ x > k$	$\frac{\alpha k}{\alpha-1},\ \alpha > 1$	$\frac{\alpha k^2}{(\alpha-2)(\alpha-1)^2},\ \alpha > 2,$	*
$\mathrm{Pa}\,(k,\alpha),\ k > 0,\ \alpha > 0$				