TAMS46: Probability Theory (Second Course) | Provkod: TEN1 | 08 January 2021, 08:00-12:00

Examiner: Xiangfeng Yang (013-285788). **Things allowed**: a calculator, an English-Swedish dictionary.
Scores rating (Betygsgränser): 8-11 points giving rate 3; 11.5-14.5 points giving rate 4; 15-18 points giving rate 5.

1 (3 points)

Let X be a continuous random variable with a density function $f_X(x) = \sqrt{\frac{2}{\pi}} \cdot x^2 \cdot e^{-x^2/2}$ for $x > 0$ (otherwise the density function is zero). Let $Y = X^2$. Find the density function $f_Y(y)$ of Y.

Solution. Let $Y = g(X) = X^2$, then $g^{-1}(y) = \sqrt{y}$ for $y > 0$ and $J = y^{-1/2}/2$. Then

$$f_Y(y) = f_X(g^{-1}(y)) \cdot |J| = \sqrt{\frac{2}{\pi}} \cdot y \cdot e^{-y/2} \cdot \frac{y^{-1/2}}{2} = \sqrt{\frac{1}{2\pi}} \cdot y^{1/2} \cdot e^{-y/2}, \text{ for } y > 0. \quad \text{(otherwise it is zero)}$$

□

2 (3 points)

Let $X_n \sim Ge(2/n)$ and $Y_n \sim Po(2n)$ for $n = 2, 3, 4, \ldots$.
(2.1) (1.5p) Prove that X_n/n converges in distribution as $n \to \infty$, and determine the limiting distribution.
(2.2) (1.5p) Prove that X_n/Y_n converges in distribution as $n \to \infty$, and determine the limiting distribution.

Solution. (2.1) The characteristic function of X_n/n is

$$\begin{aligned}\varphi_{X_n/n}(t) = \varphi_{X_n}(t/n) &= \frac{\frac{2}{n}e^{it/n}}{1-(1-2/n)e^{it/n}} = \frac{1}{(1-e^{it/n}) \cdot n/2 \cdot e^{-it/n} + 1} \\ &= \frac{1}{(e^{-it/n}-1) \cdot n/2 + 1} = \frac{1}{(-it/n + o(1/n)) \cdot n/2 + 1} \\ &\to \frac{1}{-it/2+1} = \varphi_{Exp(1/2)}(t).\end{aligned}$$

(2.2) The characteristic function of $Y_n/(2n)$ is

$$\varphi_{Y_n/(2n)}(t) = \varphi_{Y_n}(t/(2n)) = \exp\{2n(e^{it/(2n)} - 1)\} = \exp\{2n(it/(2n) + o(1/n))\} \to e^{it} = \varphi_1(t).$$

Therefore $Y_n/(2n)$ converges to 1. Therefore, it is from the Cramér-Slutsky's theorem that,

$$\frac{X_n}{Y_n} = \frac{X_n/(2n)}{Y_n/(2n)} \to \frac{\frac{1}{2} \cdot Exp(1/2)}{1} = \frac{1}{2} \cdot Exp(1/2).$$

□

3 (3 points)

Let $N \sim Fs(p)$, and let $X_1, X_2, \ldots$ be independent $Exp(1)$ random variables which are independent of N. Find the moment generating function $\psi_{S_N}(t)$ of $S_N = X_1 + X_2 + \ldots + X_N$.

Solution. It is from Theorem 6.3 of the book that the moment generating function is, with $q = 1 - p$,

$$\psi_{S_N}(t) = g_N(\psi_X(t)) = \frac{p/(1-t)}{1 - q/(1-t)} = \frac{1}{1-t/p} = \psi_{Exp(1/p)}(t).$$

□

4 (3 points)

Let $X_1, X_2, \ldots, X_n$ be independent random variables with a common distribution function $F(X)$ (that is, $F(x) = P(X_i \le x)$). Let $X_{(1)} = \min\{X_1, X_2, \ldots, X_n\}$ and $X_{(n)} = \max\{X_1, X_2, \ldots, X_n\}$. Find the joint distribution function $F_{X_{(1)},X_{(n)}}(x, y)$ of $(X_{(1)}, X_{(n)})$.

Solution. First,

$$P(X_{(1)} > x, X_{(n)} \le y) = \begin{cases} P(x < X_k \le y, \text{ for } k = 1, 2, \ldots, n) = (F(y) - F(x))^n, & \text{if } x < y, \\ 0, & \text{if } x \ge y. \end{cases}$$

Therefore,

$$\begin{aligned} F_{X_{(1)},X_{(n)}}(x, y) &= F_{X_{(n)}}(y) - P(X_{(1)} > x, X_{(n)} \le y) \\ &= \begin{cases} F(y)^n - (F(y) - F(x))^n, & \text{if } x < y, \\ F(y)^n, & \text{if } x \ge y. \end{cases} \end{aligned}$$

□

5 (3 points)

Let $N \sim Ge(p)$ and $X = (-1)^N$. Compute the mean $E(X)$ and the variance $Var(X)$.

Solution.

$$\begin{aligned} E(X) &= E(E(X|N)) = \sum_{k=0}^{\infty} E(X|N = k) \cdot P(N = k) = \sum_{k=0}^{\infty} E((-1)^k|N = k) \cdot P(N = k) = \sum_{k=0}^{\infty} (-1)^k pq^k \\ &= p \sum_{k=0}^{\infty} (-q)^k = p/(1 + q) = p/(1 + (1 - p)) = p/(2 - p). \end{aligned}$$

$$Var(X) = E(X^2) - (E(X))^2 = 1 - (p/(2 - p))^2 = \frac{4 - 4p}{(2 - p)^2},$$

where $E(X^2) = E(E(X^2|N)) = \sum_{k=0}^{\infty} (-1)^{2k} \cdot P(N = k) = \sum_{k=0}^{\infty} P(N = k) = 1.$ □

6 (3 points)

(6.1) (1.5p) Let $X_1 \sim N(\mu_1, \sigma_1^2)$ and $X_2 \sim N(\mu_2, \sigma_2^2)$ be two independent normal random variables. Prove that $X_1 + X_2$ is also a normal random variable.
(6.2) (1.5p) Let $X_3 \sim N(\mu_3, \sigma_3^2)$ and $X_4 \sim N(\mu_4, \sigma_4^2)$ be two normal random variables (which might not be independent). Does $X_3 + X_4$ have to be a normal random variable? If yes, then prove it. If not, then construct a counterexample. (Hint: note that any constant c should be regarded as a normal random variable with mean c and variance 0.)

Solution. (6.1) The characteristic functions of X_1 and X_2 are (see Appendix B):

$$\varphi_{X_1}(t) = e^{i\mu_1 t - \sigma_1^2 t^2/2} \quad \text{and} \quad \varphi_{X_2}(t) = e^{i\mu_2 t - \sigma_2^2 t^2/2}.$$

It is from the independence that

$$\varphi_{X_1+X_2}(t) = E(e^{it(X_1+X_2)}) = \varphi_{X_1}(t) \cdot \varphi_{X_2}(t) = e^{i(\mu_1+\mu_2)t - (\sigma_1^2+\sigma_2^2)t^2/2}.$$

Therefore, $X_1 + X_2 \sim N(\mu_1 + \mu_2, \sigma_1^2 + \sigma_2^2)$.
(6.2) $X_3 + X_4$ does NOT have to be normal. A counterexample is: let $X_3 \sim N(0, 1)$ and Z be independent with $P(Z = -1) = P(Z = 1) = 1/2$, and define $X_4 = Z \cdot X_3$. Then $X_4 \sim N(0, 1)$: why? because

$$\begin{aligned} P(X_4 \le x) &= P(Z \cdot X_3 \le x) = \frac{1}{2} P(X_3 \le x) + \frac{1}{2} P(-X_3 \le x) \\ &= \frac{1}{2} P(N(0, 1) \le x) + \frac{1}{2} P(-N(0, 1) \le x) \quad (\text{ note that } P(-N(0, 1) \le x) = P(N(0, 1) \le x)) \\ &= P(N(0, 1) \le x). \end{aligned}$$

Now one can see that $X_3 + X_4$ is NOT a normal random variable since:

$$P(X_3 + X_4 = 0) = P(X_3 \cdot (1 + Z) = 0) = P(Z = -1) = 1/2.$$

$\square$

Discrete Distributions

Followingis a list of discrete distributions, abbreviations, their probability functions, means, variances, and characteristic functions. An asterisk (*) indicates that the expression is too complicated to present here; in some cases a closed formula does not even exist.

Distribution, notation	Probability function	$E\,X$	$\mathrm{Var}\,X$	$\varphi_X(t)$
One point $\delta(a)$	$p(a) = 1$	a	0	e^{ita}
Symmetric Bernoulli	$p(-1) = p(1)=\frac{1}{2}$	0	1	$\cos t$
Bernoulli $\mathrm{Be}(p)$, $0 \le p \le 1$	$p(0) = q,\ p(1) = p;\ q = 1 - p$	p	pq	$q + pe^{it}$
Binomial $\mathrm{Bin}(n, p)$, $n = 1, 2, \ldots$, $0 \le p \le 1$	$p(k) = \binom{n}{k} p^k q^{n-k},\ k = 0, 1, \ldots, n;\ q = 1 - p$	np	npq	$(q + pe^{it})^n$
Geometric $\mathrm{Ge}(p)$, $0 \le p \le 1$	$p(k) = pq^k,\ k = 0, 1, 2, \ldots;\ q = 1 - p$	$\frac{q}{p}$	$\frac{q}{p^2}$	$\frac{p}{1 - qe^{it}}$
First success $\mathrm{Fs}(p)$, $0 \le p \le 1$	$p(k) = pq^{k-1},\ k = 1, 2, \ldots;\ q = 1 - p$	$\frac{1}{p}$	$\frac{q}{p^2}$	$\frac{pe^{it}}{1 - qe^{it}}$
Negative binomial $\mathrm{NBin}(n, p)$, $n = 1, 2, 3, \ldots$, $0 \le p \le 1$	$p(k) = \binom{n+k-1}{k} p^n q^k,\ k = 0, 1, 2, \ldots;$ $q = 1 - p$	$n\frac{q}{p}$	$n\frac{q}{p^2}$	$(\frac{p}{1-qe^{it}})^n$
Poisson $\mathrm{Po}(m)$, $m > 0$	$p(k) = e^{-m}\frac{m^k}{k!},\ k = 0, 1, 2, \ldots$	m	m	$e^{m(e^{it}-1)}$
Hypergeometric $H(N, n, p)$, $n = 0, 1, \ldots, N$, $N = 1, 2, \ldots$, $p = 0, \frac{1}{N}, \frac{2}{N}, \ldots, 1$	$p(k) = \frac{\binom{Np}{k}\binom{Nq}{n-k}}{\binom{N}{n}},\ k = 0, 1, \ldots, Np;$ $q = 1 - p;$ $n - k = 0, \ldots, Nq$	np	$npq\frac{N-n}{N-1}$	*

Continuous Distributions

Following is a list of some continuous distributions, abbreviations, their densities, means, variances, and characteristic functions. An asterisk (*) indicates that the expression is too complicated to present here; in some cases a closed formula does not even exist.

Distribution, notation	Density	$E\,X$	$\operatorname{Var} X$	$\varphi_X(t)$
Uniform/Rectangular				
$U(a,b)$	$f(x) = \frac{1}{b-a},\ a < x < b$	$\frac{1}{2}(a+b)$	$\frac{1}{12}(b-a)^2$	$\frac{e^{itb}-e^{ita}}{it(b-a)}$
$U(0,1)$	$f(x) = 1,\ 0 < x < 1$	$\frac{1}{2}$	$\frac{1}{12}$	$\frac{e^{it}-1}{it}$
$U(-1,1)$	$f(x) = \frac{1}{2},\ \lvert x\rvert < 1$	0	$\frac{1}{3}$	$\frac{\sin t}{t}$
Triangular				
$\mathrm{Tri}(a,b)$	$f(x) = \frac{2}{b-a}\left(1 - \frac{2}{b-a}\left\lvert x - \frac{a+b}{2}\right\rvert\right)$, $a < x < b$	$\frac{1}{2}(a+b)$	$\frac{1}{24}(b-a)^2$	$\left(\frac{e^{itb/2}-e^{ita/2}}{\frac{1}{2}it(b-a)}\right)^2$
$\mathrm{Tri}(-1,1)$	$f(x) = 1 - \lvert x\rvert,\ \lvert x\rvert < 1$	0	$\frac{1}{6}$	$\left(\frac{\sin\frac{t}{2}}{\frac{t}{2}}\right)^2$
Exponential $\mathrm{Exp}(a),\ a > 0$	$f(x) = \frac{1}{a}\,e^{-x/a},\ x > 0$	a	a^2	$\frac{1}{1-ait}$
Gamma $\Gamma(p,a),\ a > 0,\ p > 0$	$f(x) = \frac{1}{\Gamma(p)}\,x^{p-1}\,\frac{1}{a^p}\,e^{-x/a},\ x > 0$	pa	pa^2	$\frac{1}{(1-ait)^p}$
Chi-square $\chi^2(n),\ n = 1,2,3,\ldots$	$f(x) = \frac{1}{\Gamma(\frac{n}{2})}\,x^{\frac{1}{2}n-1}\left(\frac{1}{2}\right)^{n/2}e^{-x/2},\ x > 0$	n	$2n$	$\frac{1}{(1-2it)^{n/2}}$
Laplace $L(a),\ a > 0$	$f(x) = \frac{1}{2a}e^{-\lvert x\rvert/a},\ -\infty < x < \infty$	0	$2a^2$	$\frac{1}{1+a^2t^2}$
Beta $\beta(r,s),\ r,s > 0$	$f(x) = \frac{\Gamma(r+s)}{\Gamma(r)\Gamma(s)}\,x^{r-1}(1-x)^{s-1}$, $0 < x < 1$	$\frac{r}{r+s}$	$\frac{rs}{(r+s)^2(r+s+1)}$	*

Continuous Distributions (continued)

Distribution, notation	Density	$E X$	$\operatorname{Var} X$	$\varphi_X(t)$
Weibull $W(\alpha,\beta)$, $\alpha,\beta>0$	$f(x)=\frac{1}{\alpha\beta}x^{(1/\beta)-1}\,e^{-x^{1/\beta}/\alpha},\ x>0$	$\alpha^\beta\,\Gamma(\beta+1)$	$a^{2\beta}\big(\Gamma(2\beta+1) -\Gamma(\beta+1)^2\big)$	*
Rayleigh $\mathrm{Ra}\,(\alpha)$, $\alpha>0$	$f(x)=\frac{2}{\alpha}\,x\,e^{-x^2/\alpha},\ x>0$	$\frac{1}{2}\sqrt{\pi\alpha}$	$\alpha(1-\frac{1}{4}\pi)$	*
Normal $N(\mu,\sigma^2)$, $-\infty<\mu<\infty$, $\sigma>0$	$f(x)=\frac{1}{\sigma\sqrt{2\pi}}\,e^{-\frac{1}{2}(x-\mu)^2/\sigma^2},$ $-\infty<x<\infty$	μ	σ^2	$e^{i\mu t-\frac{1}{2}t^2\sigma^2}$
$N(0,1)$	$f(x)=\dfrac{1}{\sqrt{2\pi}}\,e^{-x^2/2},\ -\infty<x<\infty$	0	1	$e^{-t^2/2}$
Log-normal $LN(\mu,\sigma^2)$, $-\infty<\mu<\infty,\ \sigma>0$	$f(x)=\dfrac{1}{\sigma x\sqrt{2\pi}}\,e^{-\frac{1}{2}(\log x-\mu)^2/\sigma^2},\ x>0$	$e^{\mu+\frac{1}{2}\sigma^2}$	$e^{2\mu}\big(e^{2\sigma^2}-e^{\sigma^2}\big)$	*
(Student's) t $t(n)$, $n=1,2,\ldots$	$f(x)=\frac{\Gamma(\frac{n+1}{2})}{\sqrt{\pi n}\Gamma(\frac{n}{2})}\cdot d\frac{1}{(1+\frac{x^2}{n})^{(n+1)/2}},$ $-\infty<x<\infty$	0	$\dfrac{n}{n-2}$, $n>2$	*
(Fisher's) F $F(m,n)$, $m,n=1,2,\ldots$	$f(x)=\frac{\Gamma(\frac{m+n}{2})(\frac{m}{n})^{m/2}}{\Gamma(\frac{m}{2})\Gamma(\frac{n}{2})}\cdot\frac{x^{m/2-1}}{(1+\frac{mx}{n})^{(m+n)/2}},$ $x>0$	$\frac{n}{n-2}$, $n>2$	$\frac{n^2(m+2)}{m(n-2)(n-4)}-\left(\frac{n}{n-2}\right)^2$, $n>4$	*

Continuous Distributions (continued)

Distribution, notation	Density	$E\,X$	$\operatorname{Var} X$	$\varphi_X(t)$
Cauchy				
$C(m,a)$	$f(x) = \frac{1}{\pi} \cdot \frac{a}{a^2+(x-m)^2},\ -\infty < x < \infty$	$\nexists$	$\nexists$	$e^{imt-a\lvert t\rvert}$
$C(0,1)$	$f(x) = \frac{1}{\pi} \cdot \frac{1}{1+x^2},\ -\infty < x < \infty$	$\nexists$	$\nexists$	$e^{-\lvert t\rvert}$
Pareto $\mathrm{Pa}\,(k,\alpha),\ k>0,\ \alpha>0$	$f(x) = \frac{\alpha k^\alpha}{x^{\alpha+1}},\ x > k$	$\frac{\alpha k}{\alpha-1},\ \alpha > 1$	$\frac{\alpha k^2}{(\alpha-2)(\alpha-1)^2},\ \alpha > 2,$	*