Examiner: Xiangfeng Yang (013-285788). **Things allowed**: a calculator, an English-Swedish dictionary.
Scores rating (Betygsgränser): 8-11 points giving rate 3; 11.5-14.5 points giving rate 4; 15-18 points giving rate 5.

1 (3 points)

Let X be a continuous random variable with a density function $f_X(x) = \sqrt{\frac{2}{\pi}} \cdot x^2 \cdot e^{-x^2/2}$ for $x > 0$ (otherwise the density function is zero). Let $Y = X^2$. Find the density function $f_Y(y)$ of Y.

2 (3 points)

Let $X_n \sim Ge(2/n)$ and $Y_n \sim Po(2n)$ for $n = 2, 3, 4, \ldots$.
(2.1) (1.5p) Prove that X_n/n converges in distribution as $n \to \infty$, and determine the limiting distribution.
(2.2) (1.5p) Prove that X_n/Y_n converges in distribution as $n \to \infty$, and determine the limiting distribution.

3 (3 points)

Let $N \sim Fs(p)$, and let $X_1, X_2, \ldots$ be independent $Exp(1)$ random variables which are independent of N. Find the moment generating function $\psi_{S_N}(t)$ of $S_N = X_1 + X_2 + \ldots + X_N$.

4 (3 points)

Let $X_1, X_2, \ldots, X_n$ be independent random variables with a common distribution function $F(X)$ (that is, $F(x) = P(X_i \leq x)$). Let $X_{(1)} = \min\{X_1, X_2, \ldots, X_n\}$ and $X_{(n)} = \max\{X_1, X_2, \ldots, X_n\}$. Find the joint distribution function $F_{X_{(1)},X_{(n)}}(x, y)$ of $(X_{(1)}, X_{(n)})$.

5 (3 points)

Let $N \sim Ge(p)$ and $X = (-1)^N$. Compute the mean $E(X)$ and the variance $Var(X)$.

6 (3 points)

(6.1) (1.5p) Let $X_1 \sim N(\mu_1, \sigma_1^2)$ and $X_2 \sim N(\mu_2, \sigma_2^2)$ be two independent normal random variables. Prove that $X_1 + X_2$ is also a normal random variable.
(6.2) (1.5p) Let $X_3 \sim N(\mu_3, \sigma_3^2)$ and $X_4 \sim N(\mu_4, \sigma_4^2)$ be two normal random variables (which might not be independent). Does $X_3 + X_4$ have to be a normal random variable? If yes, then prove it. If not, then construct a counterexample. (Hint: note that any constant c should be regarded as a normal random variable with mean c and variance 0.)

Discrete Distributions

Followingis a list of discrete distributions, abbreviations, their probability functions, means, variances, and characteristic functions. An asterisk (*) indicates that the expression is too complicated to present here; in some cases a closed formula does not even exist.

Distribution, notation	Probability function	$E\,X$	$\mathrm{Var}\,X$	$\varphi_X(t)$
One point $\delta(a)$	$p(a)=1$	a	0	e^{ita}
Symmetric Bernoulli	$p(-1)=p(1)=\frac{1}{2}$	0	1	$\cos t$
Bernoulli $\mathrm{Be}(p)$, $0\le p\le 1$	$p(0)=q,\ p(1)=p;\ q=1-p$	p	pq	$q+pe^{it}$
Binomial $\mathrm{Bin}(n,p)$, $n=1,2,\ldots$, $0\le p\le 1$	$p(k)=\binom{n}{k}p^kq^{n-k},\ k=0,1,\ldots,n;\ q=1-p$	np	npq	$(q+pe^{it})^n$
Geometric $\mathrm{Ge}(p)$, $0\le p\le 1$	$p(k)=pq^k,\ k=0,1,2,\ldots;\ q=1-p$	$\frac{q}{p}$	$\frac{q}{p^2}$	$\frac{p}{1-qe^{it}}$
First success $\mathrm{Fs}(p)$, $0\le p\le 1$	$p(k)=pq^{k-1},\ k=1,2,\ldots;\ q=1-p$	$\frac{1}{p}$	$\frac{q}{p^2}$	$\frac{pe^{it}}{1-qe^{it}}$
Negative binomial $\mathrm{NBin}(n,p)$, $n=1,2,3,\ldots$, $0\le p\le 1$	$p(k)=\binom{n+k-1}{k}p^nq^k,\ k=0,1,2,\ldots;$ $q=1-p$	$n\frac{q}{p}$	$n\frac{q}{p^2}$	$(\frac{p}{1-qe^{it}})^n$
Poisson $\mathrm{Po}(m)$, $m>0$	$p(k)=e^{-m}\frac{m^k}{k!},\ k=0,1,2,\ldots$	m	m	$e^{m(e^{it}-1)}$
Hypergeometric $H(N,n,p)$, $n=0,1,\ldots,N$, $N=1,2,\ldots$, $p=0,\frac{1}{N},\frac{2}{N},\ldots,1$	$p(k)=\frac{\binom{Np}{k}\binom{Nq}{n-k}}{\binom{N}{n}},\quad k=0,1,\ldots,Np;$ $q=1-p;$ $n-k=0,\ldots,Nq$	np	$npq\frac{N-n}{N-1}$	*

Continuous Distributions

Following is a list of some continuous distributions, abbreviations, their densities, means, variances, and characteristic functions. An asterisk (*) indicates that the expression is too complicated to present here; in some cases a closed formula does not even exist.

Distribution, notation	Density	$E\,X$	$\operatorname{Var} X$	$\varphi_X(t)$
Uniform/Rectangular				
$U(a,b)$	$f(x)=\dfrac{1}{b-a},\ a<x<b$	$\frac{1}{2}(a+b)$	$\frac{1}{12}(b-a)^2$	$\frac{e^{itb}-e^{ita}}{it(b-a)}$
$U(0,1)$	$f(x)=1,\ 0<x<1$	$\frac{1}{2}$	$\frac{1}{12}$	$\frac{e^{it}-1}{it}$
$U(-1,1)$	$f(x)=\frac{1}{2},\ \lvert x\rvert<1$	0	$\frac{1}{3}$	$\frac{\sin t}{t}$
Triangular				
$\mathrm{Tri}(a,b)$	$f(x)=\dfrac{2}{b-a}\left(1-\dfrac{2}{b-a}\left\lvert x-\dfrac{a+b}{2}\right\rvert\right)$ $a<x<b$	$\frac{1}{2}(a+b)$	$\frac{1}{24}(b-a)^2$	$\left(\frac{e^{itb/2}-e^{ita/2}}{\frac{1}{2}it(b-a)}\right)^2$
$\mathrm{Tri}(-1,1)$	$f(x)=1-\lvert x\rvert,\ \lvert x\rvert<1$	0	$\frac{1}{6}$	$\left(\frac{\sin\frac{t}{2}}{\frac{t}{2}}\right)^2$
Exponential $\mathrm{Exp}(a),\ a>0$	$f(x)=\dfrac{1}{a}\,e^{-x/a},\ x>0$	a	a^2	$\dfrac{1}{1-ait}$
Gamma $\Gamma(p,a),\ a>0,\ p>0$	$f(x)=\dfrac{1}{\Gamma(p)}\,x^{p-1}\,\dfrac{1}{a^p}\,e^{-x/a},\ x>0$	pa	pa^2	$\dfrac{1}{(1-ait)^p}$
Chi-square $\chi^2(n),\ n=1,2,3,\ldots$	$f(x)=\dfrac{1}{\Gamma(\frac{n}{2})}\,x^{\frac{1}{2}n-1}\left(\frac{1}{2}\right)^{n/2}e^{-x/2},\ x>0$	n	$2n$	$\dfrac{1}{(1-2it)^{n/2}}$
Laplace $L(a),\ a>0$	$f(x)=\dfrac{1}{2a}e^{-\lvert x\rvert/a},\ -\infty<x<\infty$	0	$2a^2$	$\dfrac{1}{1+a^2t^2}$
Beta $\beta(r,s),\ r,s>0$	$f(x)=\dfrac{\Gamma(r+s)}{\Gamma(r)\Gamma(s)}\,x^{r-1}(1-x)^{s-1},$ $0<x<1$	$\dfrac{r}{r+s}$	$\dfrac{rs}{(r+s)^2(r+s+1)}$	*

Continuous Distributions (continued)

Distribution, notation	Density	$E\,X$	$\mathrm{Var}\,X$	$\varphi_X(t)$
Weibull $W(\alpha,\beta)$, $\alpha,\beta>0$	$f(x)=\frac{1}{\alpha\beta}x^{(1/\beta)-1}\,e^{-x^{1/\beta}/\alpha},\ x>0$	$\alpha^{\beta}\,\Gamma(\beta+1)$	$a^{2\beta}\big(\Gamma(2\beta+1) -\Gamma(\beta+1)^2\big)$	*
Rayleigh $\mathrm{Ra}\,(\alpha)$, $\alpha>0$	$f(x)=\frac{2}{\alpha}\,x\,e^{-x^2/\alpha},\ x>0$	$\frac{1}{2}\sqrt{\pi\alpha}$	$\alpha(1-\frac{1}{4}\pi)$	*
Normal $N(\mu,\sigma^2)$, $-\infty<\mu<\infty$, $\sigma>0$	$f(x)=\frac{1}{\sigma\sqrt{2\pi}}\,e^{-\frac{1}{2}(x-\mu)^2/\sigma^2},$ $-\infty<x<\infty$	μ	σ^2	$e^{i\mu t-\frac{1}{2}t^2\sigma^2}$
$N(0,1)$	$f(x)=\dfrac{1}{\sqrt{2\pi}}\,e^{-x^2/2},\ -\infty<x<\infty$	0	1	$e^{-t^2/2}$
Log-normal $LN(\mu,\sigma^2)$, $-\infty<\mu<\infty,\ \sigma>0$	$f(x)=\dfrac{1}{\sigma x\sqrt{2\pi}}\,e^{-\frac{1}{2}(\log x-\mu)^2/\sigma^2},\ x>0$	$e^{\mu+\frac{1}{2}\sigma^2}$	$e^{2\mu}\big(e^{2\sigma^2}-e^{\sigma^2}\big)$	*
(Student's) t $t(n)$, $n=1,2,\dots$	$f(x)=\frac{\Gamma(\frac{n+1}{2})}{\sqrt{\pi n}\Gamma(\frac{n}{2})}\cdot d\frac{1}{(1+\frac{x^2}{n})^{(n+1)/2}},$ $-\infty<x<\infty$	0	$\dfrac{n}{n-2}$, $n>2$	*
(Fisher's) F $F(m,n)$, $m,n=1,2,\dots$	$f(x)=\frac{\Gamma(\frac{m+n}{2})(\frac{m}{n})^{m/2}}{\Gamma(\frac{m}{2})\Gamma(\frac{n}{2})}\cdot\frac{x^{m/2-1}}{(1+\frac{mx}{n})^{(m+n)/2}},$ $x>0$	$\frac{n}{n-2}$, $n>2$	$\frac{n^2(m+2)}{m(n-2)(n-4)}-\Big(\frac{n}{n-2}\Big)^2,$ $n>4$	*

Continuous Distributions (continued)

Distribution, notation	Density	$E\,X$	$\operatorname{Var} X$	$\varphi_X(t)$
Cauchy				
$C(m,a)$	$f(x) = \frac{1}{\pi} \cdot \frac{a}{a^2+(x-m)^2}, \ -\infty < x < \infty$	$\nexists$	$\nexists$	$e^{imt-a\|t\|}$
$C(0,1)$	$f(x) = \dfrac{1}{\pi} \cdot \dfrac{1}{1+x^2}, \ -\infty < x < \infty$	$\nexists$	$\nexists$	$e^{-\|t\|}$
Pareto	$f(x) = \dfrac{\alpha k^\alpha}{x^{\alpha+1}}, \ x > k$	$\dfrac{\alpha k}{\alpha - 1}, \ \alpha > 1$	$\dfrac{\alpha k^2}{(\alpha-2)(\alpha-1)^2}, \ \alpha > 2,$	*
$\mathrm{Pa}\,(k,\alpha),\ k > 0,\ \alpha > 0$				