TAMS46: Probability Theory (Second Course) | Provkod: TEN1 | 29 October 2022, 14:00-18:00

Examiner: Xiangfeng Yang (013-285788). **Things allowed**: a calculator, a self-written A4 paper (two sides).
Scores rating (Betygsgränser): 8-11 points giving rate 3; 11.5-14.5 points giving rate 4; 15-18 points giving rate 5.
Notation: 'A random variable X is distributed as...' is written as '$X \in$ ... or $X \sim$ ... '

1 (3 points)

(1.1) (1p) Let X be a continuous one-dimensional random variable with a probability density function $f_X(x), x \in \mathbb{R}$. Define $Y = X^2$, find the probability density function $f_Y(y)$ of Y.
(1.2) (2p) Let X_1 and X_2 be independent $Exp(1)$-distributed random variables. Find the density function of $\frac{X_1}{X_1+X_2}$.

2 (3 points)

Let X be a Poisson random variable with a random parameter M as follows:

$$X|M = m \sim Po(m), \qquad \text{with } M \sim Exp(1).$$

(2.1) (1p) Find the mean $E(X)$ of X.
(2.2) (1p) Find $E(X \cdot M)$.
(2.3) (1p) Find the probability $P(X = 1)$.

3 (3 points)

Suppose that X is a random variable such that

$$E(X^n) = \frac{1}{4} + 2^{n-1}, \quad n = 1, 2, \ldots.$$

(3.1) (2p) Find the moment generating function $\psi_X(t)$ of X.
(3.2) (1p) Determine the probabilities $P(X = k)$ for $k = 0, 1, 2, \ldots$.

4 (3 points)

Suppose that X_1, X_2 and X_3 are independent $U(0,1)$ random variables, and let $(X_{(1)}, X_{(2)}, X_{(3)})$ be the corresponding order statistic.
(4.1) (2p) Find the conditional probability density function $f_{X_{(3)}|X_{(1)}=y_1}(y_3)$ of $X_{(3)}$ given $X_{(1)} = y_1$.
(4.2) (1p) Find the probability $P(X_{(3)} \geq 2X_{(1)})$.

5 (3 points)

Let $\mathbf{X} = (X_1, X_2)'$ be a two dimensional normal random variable $\mathbf{X} \sim N(\mu, \Lambda)$, where the mean vector is $\mu = (0,0)'$ and the covariance matrix is $= \begin{pmatrix} 3 & 1 \\ 1 & 2 \end{pmatrix}$. Define a new two dimensional random variable as $\mathbf{Y} = (Y_1, Y_2)'$ with $Y_1 = X_1 + X_2$ and $Y_2 = X_1 - X_2$.
(5.1) (1.5p) Find the distribution of $\mathbf{Y}$.
(5.2) (1.5p) Find the conditional distribution of Y_2 given $Y_1 = 1$.

6 (3 points)

Let $X_1, X_2, \ldots$ be i.i.d. (independent and identically distributed) random variables with finite mean $\mu = E(X_i) \neq 0$ and finite variance $\sigma^2 = Var(X_i) \neq 0$. Let $S_n = X_1 + X_2 + \ldots + X_n$ for $n \geq 1$.
(6.1) (1p)

Does $\dfrac{S_n - n\mu}{S_n + n\mu}$ converge in probability? If yes, then find the limit; if no, then explain why.

(6.2) (2p)

Does $\sqrt{n} \cdot \dfrac{S_n - n\mu}{S_n + n\mu}$ converge in distribution? If yes, then find the limit; if no, then explain why.

Discrete Distributions

Followingis a list of discrete distributions, abbreviations, their probability functions, means, variances, and characteristic functions. An asterisk (*) indicates that the expression is too complicated to present here; in some cases a closed formula does not even exist.

Distribution, notation	Probability function	$E\,X$	$\mathrm{Var}\,X$	$\varphi_X(t)$
One point $\delta(a)$	$p(a) = 1$	a	0	e^{ita}
Symmetric Bernoulli	$p(-1) = p(1) = \frac{1}{2}$	0	1	$\cos t$
Bernoulli $\mathrm{Be}(p)$, $0 \le p \le 1$	$p(0) = q,\ p(1) = p;\ q = 1 - p$	p	pq	$q + pe^{it}$
Binomial $\mathrm{Bin}(n, p)$, $n = 1, 2, \ldots$, $0 \le p \le 1$	$p(k) = \binom{n}{k} p^k q^{n-k},\ k = 0, 1, \ldots, n;\ q = 1 - p$	np	npq	$(q + pe^{it})^n$
Geometric $\mathrm{Ge}(p)$, $0 \le p \le 1$	$p(k) = pq^k,\ k = 0, 1, 2, \ldots;\ q = 1 - p$	$\frac{q}{p}$	$\frac{q}{p^2}$	$\frac{p}{1 - qe^{it}}$
First success $\mathrm{Fs}(p)$, $0 \le p \le 1$	$p(k) = pq^{k-1},\ k = 1, 2, \ldots;\ q = 1 - p$	$\frac{1}{p}$	$\frac{q}{p^2}$	$\frac{pe^{it}}{1 - qe^{it}}$
Negative binomial $\mathrm{NBin}(n, p)$, $n = 1, 2, 3, \ldots$, $0 \le p \le 1$	$p(k) = \binom{n+k-1}{k} p^n q^k,\ k = 0, 1, 2, \ldots;$ $q = 1 - p$	$n\frac{q}{p}$	$n\frac{q}{p^2}$	$(\frac{p}{1 - qe^{it}})^n$
Poisson $\mathrm{Po}(m)$, $m > 0$	$p(k) = e^{-m}\frac{m^k}{k!},\ k = 0, 1, 2, \ldots$	m	m	$e^{m(e^{it}-1)}$
Hypergeometric $H(N, n, p)$, $n = 0, 1, \ldots, N$, $N = 1, 2, \ldots$, $p = 0, \frac{1}{N}, \frac{2}{N}, \ldots, 1$	$p(k) = \dfrac{\binom{Np}{k}\binom{Nq}{n-k}}{\binom{N}{n}},\quad k = 0, 1, \ldots, Np;$ $q = 1 - p;$ $n - k = 0, \ldots, Nq$	np	$npq\frac{N-n}{N-1}$	*

Continuous Distributions

Following is a list of some continuous distributions, abbreviations, their densities, means, variances, and characteristic functions. An asterisk (*) indicates that the expression is too complicated to present here; in some cases a closed formula does not even exist.

Distribution, notation	Density	$E\,X$	$\mathrm{Var}\,X$	$\varphi_X(t)$
Uniform/Rectangular				
$U(a,b)$	$f(x)=\dfrac{1}{b-a},\ a<x<b$	$\frac{1}{2}(a+b)$	$\frac{1}{12}(b-a)^2$	$\frac{e^{itb}-e^{ita}}{it(b-a)}$
$U(0,1)$	$f(x)=1,\ 0<x<1$	$\frac{1}{2}$	$\frac{1}{12}$	$\frac{e^{it}-1}{it}$
$U(-1,1)$	$f(x)=\frac{1}{2},\ \lvert x\rvert<1$	0	$\frac{1}{3}$	$\frac{\sin t}{t}$
Triangular				
$\mathrm{Tri}(a,b)$	$f(x)=\dfrac{2}{b-a}\left(1-\dfrac{2}{b-a}\left\lvert x-\dfrac{a+b}{2}\right\rvert\right)$, $a<x<b$	$\frac{1}{2}(a+b)$	$\frac{1}{24}(b-a)^2$	$\left(\frac{e^{itb/2}-e^{ita/2}}{\frac{1}{2}it(b-a)}\right)^2$
$\mathrm{Tri}(-1,1)$	$f(x)=1-\lvert x\rvert,\ \lvert x\rvert<1$	0	$\frac{1}{6}$	$\left(\frac{\sin\frac{t}{2}}{\frac{t}{2}}\right)^2$
Exponential $\mathrm{Exp}(a),\ a>0$	$f(x)=\dfrac{1}{a}\,e^{-x/a},\ x>0$	a	a^2	$\dfrac{1}{1-ait}$
Gamma $\Gamma(p,a),\ a>0,\ p>0$	$f(x)=\dfrac{1}{\Gamma(p)}\,x^{p-1}\,\dfrac{1}{a^p}\,e^{-x/a},\ x>0$	pa	pa^2	$\dfrac{1}{(1-ait)^p}$
Chi-square $\chi^2(n),\ n=1,2,3,\dots$	$f(x)=\dfrac{1}{\Gamma(\frac{n}{2})}\,x^{\frac{1}{2}n-1}\left(\frac{1}{2}\right)^{n/2}e^{-x/2},\ x>0$	n	$2n$	$\dfrac{1}{(1-2it)^{n/2}}$
Laplace $L(a),\ a>0$	$f(x)=\dfrac{1}{2a}e^{-\lvert x\rvert/a},\ -\infty<x<\infty$	0	$2a^2$	$\dfrac{1}{1+a^2t^2}$
Beta $\beta(r,s),\ r,s>0$	$f(x)=\dfrac{\Gamma(r+s)}{\Gamma(r)\Gamma(s)}\,x^{r-1}(1-x)^{s-1}$, $0<x<1$	$\dfrac{r}{r+s}$	$\dfrac{rs}{(r+s)^2(r+s+1)}$	*

Continuous Distributions (continued)

Distribution, notation	Density	$E\,X$	$\mathrm{Var}\,X$	$\varphi_X(t)$
Weibull $W(\alpha,\beta)$, $\alpha,\beta>0$	$f(x)=\frac{1}{\alpha\beta}x^{(1/\beta)-1}\,e^{-x^{1/\beta}/\alpha},\ x>0$	$\alpha^\beta\,\Gamma(\beta+1)$	$a^{2\beta}\big(\Gamma(2\beta+1) -\Gamma(\beta+1)^2\big)$	*
Rayleigh $\mathrm{Ra}\,(\alpha)$, $\alpha>0$	$f(x)=\frac{2}{\alpha}\,x\,e^{-x^2/\alpha},\ x>0$	$\frac{1}{2}\sqrt{\pi\alpha}$	$\alpha(1-\frac{1}{4}\pi)$	*
Normal $N(\mu,\sigma^2)$, $-\infty<\mu<\infty$, $\sigma>0$	$f(x)=\frac{1}{\sigma\sqrt{2\pi}}\,e^{-\frac{1}{2}(x-\mu)^2/\sigma^2},$ $-\infty<x<\infty$	μ	σ^2	$e^{i\mu t-\frac{1}{2}t^2\sigma^2}$
$N(0,1)$	$f(x)=\dfrac{1}{\sqrt{2\pi}}\,e^{-x^2/2},\ -\infty<x<\infty$	0	1	$e^{-t^2/2}$
Log-normal $LN(\mu,\sigma^2)$, $-\infty<\mu<\infty,\ \sigma>0$	$f(x)=\dfrac{1}{\sigma x\sqrt{2\pi}}\,e^{-\frac{1}{2}(\log x-\mu)^2/\sigma^2},\ x>0$	$e^{\mu+\frac{1}{2}\sigma^2}$	$e^{2\mu}\big(e^{2\sigma^2}-e^{\sigma^2}\big)$	*
(Student's) t $t(n)$, $n=1,2,\ldots$	$f(x)=\frac{\Gamma(\frac{n+1}{2})}{\sqrt{\pi n}\Gamma(\frac{n}{2})}\cdot d\frac{1}{(1+\frac{x^2}{n})^{(n+1)/2}},$ $-\infty<x<\infty$	0	$\dfrac{n}{n-2},\ n>2$	*
(Fisher's) F $F(m,n)$, $m,n=1,2,\ldots$	$f(x)=\frac{\Gamma(\frac{m+n}{2})(\frac{m}{n})^{m/2}}{\Gamma(\frac{m}{2})\Gamma(\frac{n}{2})}\cdot\frac{x^{m/2-1}}{(1+\frac{mx}{n})^{(m+n)/2}},$ $x>0$	$\frac{n}{n-2},$ $n>2$	$\frac{n^2(m+2)}{m(n-2)(n-4)}-\Big(\frac{n}{n-2}\Big)^2,$ $n>4$	*

Continuous Distributions (continued)

Distribution, notation	Density	$E\,X$	$\mathrm{Var}\,X$	$\varphi_X(t)$
Cauchy				
$C(m,a)$	$f(x) = \frac{1}{\pi} \cdot \frac{a}{a^2+(x-m)^2},\ -\infty < x < \infty$	$\nexists$	$\nexists$	$e^{imt-a\lvert t\rvert}$
$C(0,1)$	$f(x) = \frac{1}{\pi} \cdot \frac{1}{1+x^2},\ -\infty < x < \infty$	$\nexists$	$\nexists$	$e^{-\lvert t\rvert}$
Pareto	$f(x) = \frac{\alpha k^\alpha}{x^{\alpha+1}},\ x > k$	$\frac{\alpha k}{\alpha-1},\ \alpha > 1$	$\frac{\alpha k^2}{(\alpha-2)(\alpha-1)^2},\ \alpha > 2,$	*
$\mathrm{Pa}\,(k,\alpha),\ k > 0,\ \alpha > 0$				