Examiner: Xiangfeng Yang (013-285788). **Things allowed**: a calculator, a self-written A4 paper (two sides).
Scores rating (Betygsgränser): 8-11 points giving rate 3; 11.5-14.5 points giving rate 4; 15-18 points giving rate 5.
Notation: 'A random variable X is distributed as...' is written as '$X \in$ ... or $X \sim$ ... '

1 (3 points)

Let $X \sim U(0,1)$ and $Y \sim Exp(1)$ be independent random variables. Find the probability density function of $X+Y$.

2 (3 points)

Let $(X,Y)'$ have a joint probability density function as follows

$$f(x,y) = \begin{cases} c \cdot x \cdot y, & \text{if } 0 < y < x < 1, \\ 0, & \text{otherwise.} \end{cases}$$

(2.1) (1p) Find the value of c such that $f(x,y)$ is indeed a density function.
(2.2) (1p) Compute the conditional expectation $E(Y|X=x)$ for $0 < x < 1$.
(2.3) (1p) Compute the conditional expectation $E(X|Y=y)$ for $0 < y < 1$.

3 (3 points)

Let the probability generating function $g_{X,Y}(s,t)$ of $(X,Y)'$ be given as

$$g_{X,Y}(s,t) = E(s^X t^Y) = \exp\{(s-1) + 2(t-1) + 3(st-1)\}.$$

(3.1) (1p) Find the probability generating function $g_X(s)$ of X and $P(X=n)$ for $n \geq 0$.
(3.2) (1p) Find the probability generating function $g_Y(t)$ of Y and $P(Y=n)$ for $n \geq 0$.
(3.3) (1p) Find the probability generating function $g_{X+Y}(u)$ of $X+Y$.

4 (3 points)

Suppose that X_1, X_2, X_3 and X_4 are independent $U(0,1)$ random variables, and let $(X_{(1)}, X_{(2)}, X_{(3)}, X_{(4)})$ be the corresponding order statistic. Find the probability $P(X_{(3)} + X_{(4)} \leq 1)$.

5 (3 points)

Let X and Y be two random variables such that $X \sim N(3, 4^2)$ and $Y|X=x \sim N(10+20x, 5^2)$ (that is, the conditional distribution of Y given $X=x$ is $N(10+20x, 5^2)$). Find the mean vector $\boldsymbol{\mu}$ and the covariance matrix $\mathbf{C}$ of the two dimensional random variable $(X,Y)'$.

6 (3 points)

Let $X_n \sim Bin(n^2, 1/n)$. Use convergence of moment generating functions to show that

$$\frac{X_n - n}{\sqrt{n}} \xrightarrow{d} N(0,1), \quad \text{as } n \to \infty.$$

(Hint: moment generating function of Binomial random variable is $\psi_{Bin(n,p)}(t) = [(1-p)+pe^t]^n$, and moment generating function of standard normal random variable is $\psi_{N(0,1)}(t) = e^{t^2/2}$. You might also need to use the expansions $e^x - 1 = x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots$ and $\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} + \ldots$)

Discrete Distributions

Followingis a list of discrete distributions, abbreviations, their probability functions, means, variances, and characteristic functions. An asterisk (*) indicates that the expression is too complicated to present here; in some cases a closed formula does not even exist.

Distribution, notation	Probability function	$E\,X$	$\mathrm{Var}\,X$	$\varphi_X(t)$
One point $\delta(a)$	$p(a)=1$	a	0	e^{ita}
Symmetric Bernoulli	$p(-1)=p(1)=\frac{1}{2}$	0	1	$\cos t$
Bernoulli $\mathrm{Be}(p)$, $0\le p\le 1$	$p(0)=q,\ p(1)=p;\ q=1-p$	p	pq	$q+pe^{it}$
Binomial $\mathrm{Bin}(n,p)$, $n=1,2,\ldots,$ $0\le p\le 1$	$p(k)=\binom{n}{k}p^kq^{n-k},\ k=0,1,\ldots,n;\ q=1-p$	np	npq	$(q+pe^{it})^n$
Geometric $\mathrm{Ge}(p)$, $0\le p\le 1$	$p(k)=pq^k,\ k=0,1,2,\ldots;\ q=1-p$	$\dfrac{q}{p}$	$\dfrac{q}{p^2}$	$\dfrac{p}{1-qe^{it}}$
First success $\mathrm{Fs}(p)$, $0\le p\le 1$	$p(k)=pq^{k-1},\ k=1,2,\ldots;\ q=1-p$	$\dfrac{1}{p}$	$\dfrac{q}{p^2}$	$\dfrac{pe^{it}}{1-qe^{it}}$
Negative binomial $\mathrm{NBin}(n,p)$, $n=1,2,3,\ldots,$ $0\le p\le 1$	$p(k)=\binom{n+k-1}{k}p^nq^k,\ k=0,1,2,\ldots;$ $q=1-p$	$n\dfrac{q}{p}$	$n\dfrac{q}{p^2}$	$(\frac{p}{1-qe^{it}})^n$
Poisson $\mathrm{Po}(m)$, $m>0$	$p(k)=e^{-m}\dfrac{m^k}{k!},\ k=0,1,2,\ldots$	m	m	$e^{m(e^{it}-1)}$
Hypergeometric $H(N,n,p)$, $n=0,1,\ldots,N,$ $N=1,2,\ldots,$ $p=0,\dfrac{1}{N},\dfrac{2}{N},\ldots,1$	$p(k)=\dfrac{\binom{Np}{k}\binom{Nq}{n-k}}{\binom{N}{n}},\quad k=0,1,\ldots,Np;$ $q=1-p;$ $n-k=0,\ldots,Nq$	np	$npq\,\dfrac{N-n}{N-1}$	*

Continuous Distributions

Following is a list of some continuous distributions, abbreviations, their densities, means, variances, and characteristic functions. An asterisk (*) indicates that the expression is too complicated to present here; in some cases a closed formula does not even exist.

Distribution, notation	Density	$E\,X$	$\operatorname{Var} X$	$\varphi_X(t)$
Uniform/Rectangular				
$U(a,b)$	$f(x) = \dfrac{1}{b-a},\ a < x < b$	$\frac{1}{2}(a+b)$	$\frac{1}{12}(b-a)^2$	$\frac{e^{itb}-e^{ita}}{it(b-a)}$
$U(0,1)$	$f(x) = 1,\ 0 < x < 1$	$\frac{1}{2}$	$\frac{1}{12}$	$\frac{e^{it}-1}{it}$
$U(-1,1)$	$f(x) = \frac{1}{2},\ \lvert x\rvert < 1$	0	$\frac{1}{3}$	$\frac{\sin t}{t}$
Triangular				
$\mathrm{Tri}(a,b)$	$f(x) = \dfrac{2}{b-a}\left(1 - \dfrac{2}{b-a}\left\lvert x - \dfrac{a+b}{2}\right\rvert\right)$, $a < x < b$	$\frac{1}{2}(a+b)$	$\frac{1}{24}(b-a)^2$	$\left(\frac{e^{itb/2}-e^{ita/2}}{\frac{1}{2}it(b-a)}\right)^2$
$\mathrm{Tri}(-1,1)$	$f(x) = 1 - \lvert x\rvert,\ \lvert x\rvert < 1$	0	$\frac{1}{6}$	$\left(\frac{\sin\frac{t}{2}}{\frac{t}{2}}\right)^2$
Exponential $\mathrm{Exp}(a),\ a > 0$	$f(x) = \dfrac{1}{a}\, e^{-x/a},\ x > 0$	a	a^2	$\dfrac{1}{1-ait}$
Gamma $\Gamma(p,a),\ a > 0,\ p > 0$	$f(x) = \dfrac{1}{\Gamma(p)}\, x^{p-1}\, \dfrac{1}{a^p}\, e^{-x/a},\ x > 0$	pa	pa^2	$\dfrac{1}{(1-ait)^p}$
Chi-square $\chi^2(n),\ n = 1,2,3,\ldots$	$f(x) = \dfrac{1}{\Gamma(\frac{n}{2})}\, x^{\frac{1}{2}n-1} \left(\frac{1}{2}\right)^{n/2} e^{-x/2},\ x > 0$	n	$2n$	$\dfrac{1}{(1-2it)^{n/2}}$
Laplace $L(a),\ a > 0$	$f(x) = \dfrac{1}{2a} e^{-\lvert x\rvert/a},\ -\infty < x < \infty$	0	$2a^2$	$\dfrac{1}{1+a^2t^2}$
Beta $\beta(r,s),\ r,s > 0$	$f(x) = \dfrac{\Gamma(r+s)}{\Gamma(r)\Gamma(s)}\, x^{r-1}(1-x)^{s-1}$, $0 < x < 1$	$\dfrac{r}{r+s}$	$\dfrac{rs}{(r+s)^2(r+s+1)}$	*

Continuous Distributions (continued)

Distribution, notation	Density	$E\,X$	$\mathrm{Var}\,X$	$\varphi_X(t)$
Weibull $W(\alpha,\beta)$, $\alpha,\beta>0$	$f(x)=\frac{1}{\alpha\beta}x^{(1/\beta)-1}\,e^{-x^{1/\beta}/\alpha},\ x>0$	$\alpha^{\beta}\,\Gamma(\beta+1)$	$a^{2\beta}\big(\Gamma(2\beta+1)$ $-\Gamma(\beta+1)^2\big)$	$*$
Rayleigh $\mathrm{Ra}\,(\alpha)$, $\alpha>0$	$f(x)=\frac{2}{\alpha}\,x\,e^{-x^2/\alpha},\ x>0$	$\frac{1}{2}\sqrt{\pi\alpha}$	$\alpha(1-\frac{1}{4}\pi)$	$*$
Normal $N(\mu,\sigma^2)$, $-\infty<\mu<\infty$, $\sigma>0$	$f(x)=\frac{1}{\sigma\sqrt{2\pi}}\,e^{-\frac{1}{2}(x-\mu)^2/\sigma^2},$ $-\infty<x<\infty$	μ	σ^2	$e^{i\mu t-\frac{1}{2}t^2\sigma^2}$
$N(0,1)$	$f(x)=\dfrac{1}{\sqrt{2\pi}}\,e^{-x^2/2},\ -\infty<x<\infty$	0	1	$e^{-t^2/2}$
Log-normal $LN(\mu,\sigma^2)$, $-\infty<\mu<\infty,\ \sigma>0$	$f(x)=\dfrac{1}{\sigma x\sqrt{2\pi}}\,e^{-\frac{1}{2}(\log x-\mu)^2/\sigma^2},\ x>0$	$e^{\mu+\frac{1}{2}\sigma^2}$	$e^{2\mu}\big(e^{2\sigma^2}-e^{\sigma^2}\big)$	$*$
(Student's) t $t(n)$, $n=1,2,\ldots$	$f(x)=\frac{\Gamma(\frac{n+1}{2})}{\sqrt{\pi n}\Gamma(\frac{n}{2})}\cdot d\frac{1}{(1+\frac{x^2}{n})^{(n+1)/2}},$ $-\infty<x<\infty$	0	$\dfrac{n}{n-2},\ n>2$	$*$
(Fisher's) F $F(m,n)$, $m,n=1,2,\ldots$	$f(x)=\frac{\Gamma(\frac{m+n}{2})(\frac{m}{n})^{m/2}}{\Gamma(\frac{m}{2})\Gamma(\frac{n}{2})}\cdot\frac{x^{m/2-1}}{(1+\frac{mx}{n})^{(m+n)/2}},$ $x>0$	$\frac{n}{n-2},$ $n>2$	$\frac{n^2(m+2)}{m(n-2)(n-4)}-\left(\frac{n}{n-2}\right)^2,$ $n>4$	$*$

Continuous Distributions (continued)

Distribution, notation	Density	$E\,X$	$\mathrm{Var}\,X$	$\varphi_X(t)$
Cauchy				
$C(m,a)$	$f(x)=\frac{1}{\pi}\cdot\frac{a}{a^2+(x-m)^2},\ -\infty<x<\infty$	$\nexists$	$\nexists$	$e^{imt-a\lvert t\rvert}$
$C(0,1)$	$f(x)=\frac{1}{\pi}\cdot\frac{1}{1+x^2},\ -\infty<x<\infty$	$\nexists$	$\nexists$	$e^{-\lvert t\rvert}$
Pareto	$f(x)=\frac{\alpha k^\alpha}{x^{\alpha+1}},\ x>k$	$\frac{\alpha k}{\alpha-1},\ \alpha>1$	$\frac{\alpha k^2}{(\alpha-2)(\alpha-1)^2},\ \alpha>2,$	*
$\mathrm{Pa}\,(k,\alpha),\ k>0,\ \alpha>0$				