8.17. Bestäm den symmetriska matris som hör till följande kvadratiska former

(a) $Q_a = 4x^2 + xy - 3y^2 = (x\ y)\underline{\begin{pmatrix} 4 & \frac{1}{2} \\ \frac{1}{2} & -3 \end{pmatrix}}\begin{pmatrix} x \\ y \end{pmatrix}$

(b) $Q_b = x_1^2 + 2x_2^2 + 3x_1x_2 = (x_1\ x_2)\underline{\begin{pmatrix} 1 & 3/2 \\ 3/2 & 2 \end{pmatrix}}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

(c) $Q_c = x_1^2 + 2x_2^2 + 3x_3^2 + x_1x_3 - 3x_2x_3 =$

$= (x_1\ x_2\ x_3)\underline{\begin{pmatrix} 1 & 0 & \frac{1}{2} \\ 0 & 2 & -3/2 \\ \frac{1}{2} & -3/2 & 3 \end{pmatrix}}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$

8.18 Bestäm den kvadratiska form som hör till följande symmetriska matriser:

(a) $Q_a = (x\ y)\begin{pmatrix} 3 & 2 \\ 2 & -3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 3x^2 + 4xy - 3y^2$

(b) $Q_b = (x_1\ x_2\ x_3)\begin{pmatrix} 3 & 2 & 0 \\ 2 & -3 & 1 \\ 0 & 1 & 5 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} =$

$3x_1^2 - 3x_2^2 + 5x_3^2 + 4x_1x_2 + 2x_2x_3$

8.20 Diagonalisera följande kvadratiska former.

(a) $Q(x,y) = -3x^2 + 8xy + 3y^2$

$$A = \begin{bmatrix} -3 & 4 \\ 4 & 3 \end{bmatrix}, \quad |A - \lambda E| = 0 \Leftrightarrow \begin{vmatrix} (-3-\lambda) & 4 \\ 4 & (3-\lambda) \end{vmatrix} = 0$$

$\Leftrightarrow \lambda^2 - 25 = 0 \Leftrightarrow \lambda_{1,2} = \pm 5.$

$\lambda_1 = 5$: $\left[\begin{array}{cc|c} -8 & 4 & 0 \\ 4 & -2 & 0 \end{array}\right] \sim \begin{array}{c} \begin{array}{ccc} x_1 & x_2 & \end{array} \\ \left[\begin{array}{cc|c} 1 & -\frac{1}{2} & 0 \end{array}\right] \end{array}$

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = t\begin{pmatrix} \frac{1}{2} \\ 1 \end{pmatrix}, \; t \neq 0 \quad \Rightarrow \quad \bar{p}_1 = \frac{1}{\sqrt{5}}\begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

$\lambda_2 = -5$: $\left[\begin{array}{cc|c} 2 & 4 & 0 \\ 4 & 8 & 0 \end{array}\right] \sim \begin{array}{c} \begin{array}{ccc} x_1 & x_2 & \end{array} \\ \left[\begin{array}{cc|c} 1 & 2 & 0 \end{array}\right] \end{array}$

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = t\begin{pmatrix} -2 \\ 1 \end{pmatrix}, \; t \neq 0 \quad \Rightarrow \quad \bar{p}_2 = \frac{1}{\sqrt{5}}\begin{pmatrix} -2 \\ 1 \end{pmatrix}$$

Obs 1) $\bar{p}_1, \bar{p}_2$ är en ON bas bestående av egenvektorer till A.

2) $A = PDP^t$, $\quad P = [\bar{p}_1 | \bar{p}_2]$, $D = \begin{bmatrix} 5 & 0 \\ 0 & -5 \end{bmatrix}$

3) Variabelbyte: $\begin{pmatrix} x \\ y \end{pmatrix} = P\begin{pmatrix} u \\ v \end{pmatrix} = \frac{1}{\sqrt{5}}\begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix}$

$\underline{\circ}$ $\quad Q = 5u^2 - 5v^2$

(b) $Q(x,y,z) = 3x^2 + 3y^2 + 6z^2 - 2xy$

$$A = \begin{bmatrix} 3 & -1 & 0 \\ -1 & 3 & 0 \\ 0 & 0 & 6 \end{bmatrix}, \quad |A - \lambda E| = 0 \Leftrightarrow \begin{vmatrix} (3-\lambda) & -1 & 0 \\ -1 & (3-\lambda) & 0 \\ 0 & 0 & (6-\lambda) \end{vmatrix} = 0$$

$$\Leftrightarrow (6-\lambda)(\lambda^2 - 6\lambda + 8) = 0 \Leftrightarrow \lambda_1 = 2, \ \lambda_2 = 4, \ \lambda_3 = 6$$

$\underline{\lambda_1 = 2}$:
$$\left[\begin{array}{ccc|c} 1 & -1 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 4 & 0 \end{array}\right] \sim \begin{array}{c} \begin{array}{ccc} x_1 & x_2 & x_3 \end{array} \\ \left[\begin{array}{ccc|c} 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right] \end{array} \sim$$

$$\begin{pmatrix} x_1 \\ x_2 \\ \lambda_3 \end{pmatrix} = t \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \ t \neq 0 \Rightarrow \bar{p}_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

$\underline{\lambda_2 = 4}$:
$$\left[\begin{array}{ccc|c} -1 & -1 & 0 & 0 \\ -1 & -1 & 0 & 0 \\ 0 & 0 & 2 & 0 \end{array}\right] \sim \begin{array}{c} \begin{array}{ccc} x_1 & x_2 & x_3 \end{array} \\ \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right] \end{array} \sim$$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = t \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \ t \neq 0 \Rightarrow \bar{p}_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$$

$\underline{\lambda_3 = 6}$:
$$\left[\begin{array}{ccc|c} -3 & -1 & 0 & 0 \\ -1 & -3 & 0 & 0 \\ \cancel{0} & \cancel{0} & \cancel{0} & \cancel{0} \end{array}\right] \underset{\times(-3)}{\sim} \left[\begin{array}{ccc|c} 1 & 3 & 0 & 0 \\ 0 & 8 & 0 & 0 \end{array}\right] \sim$$

$$\sim \begin{array}{c} \begin{array}{ccc} x_1 & x_2 & x_3 \end{array} \\ \left[\begin{array}{ccc|c} 1 & 3 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{array}\right] \end{array} \sim \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = t \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \ t \neq 0 \Rightarrow \bar{p}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

Obs 1) $\bar{p}_1, \bar{p}_2, \bar{p}_3$ är en ON bas bestående av egenvektorer till A.

2) $A = PDP^t$, $P = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & -1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & \sqrt{2} \end{bmatrix}$, $D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 6 \end{bmatrix}$

3) Variabelbyte: $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = P\begin{pmatrix} u \\ v \\ w \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & \sqrt{2} \end{pmatrix}\begin{pmatrix} u \\ v \\ w \end{pmatrix}$

$\underline{\Rightarrow}$ $Q = 2u^2 + 4v^2 + 6w^2$

8.21 Bestäm den geometriska betydelsen av ekvationerna.

(a) $x^2 + xy + y^2 = 3$ (ekvationen från 8.22).

$Q = x^2 + xy + y^2$, $A = \begin{bmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & 1 \end{bmatrix}$,

$|A - \lambda E| = 0 \Leftrightarrow \begin{vmatrix} (1-\lambda) & \frac{1}{2} \\ \frac{1}{2} & (1-\lambda) \end{vmatrix} = 0 \Leftrightarrow \lambda^2 - 2\lambda + \frac{3}{4} = 0$

$\lambda_{1,2} = \frac{1}{2}, \frac{3}{2}$

$\lambda_1 = \frac{1}{2}$: $\left[\begin{array}{cc|c} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 \end{array}\right] \sim \begin{array}{cc} x_1 & x_2 \\ \hline \end{array} \left[\begin{array}{cc|c} 1 & 1 & 0 \end{array}\right] \sim$

$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = t\begin{pmatrix} -1 \\ 1 \end{pmatrix}$, $t \neq 0$ $\Rightarrow$ $\bar{p}_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} -1 \\ 1 \end{pmatrix}$

$$\underline{\lambda_2 = 3/2}: \begin{bmatrix} -1/2 & 1/2 & | & 0 \\ 1/2 & -1/2 & | & 0 \end{bmatrix} \Longleftrightarrow \begin{array}{cc} x_1 & x_2 \end{array} \atop \begin{bmatrix} 1 & -1 & | & 0 \end{bmatrix} \sim$$

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = t\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \ t \neq 0 \Rightarrow \bar{p}_2 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

<u>Obs</u> 1) $\bar{p}_1 \bar{p}_2$ är en ON bas bestående av egenvektorer till A

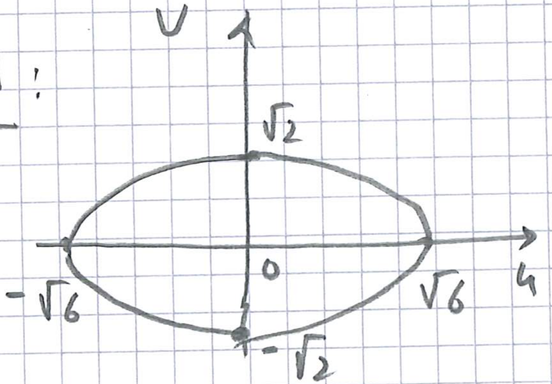
2) $A = P D P^t$, $P = \frac{1}{\sqrt{2}}\begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}$, $D = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & 3/2 \end{pmatrix}$

3) Variabelbyte: $\begin{pmatrix} x \\ y \end{pmatrix} = P\begin{pmatrix} u \\ v \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix}$

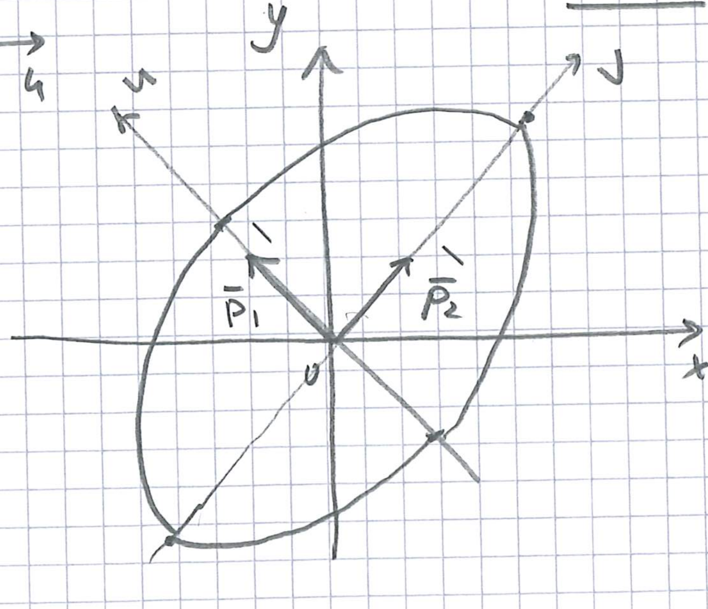
$\underline{0}$ $\quad \frac{1}{2}u^2 + \frac{3}{2}v^2 = 3 \Longleftrightarrow \frac{u^2}{(\sqrt{6})^2} + \frac{v^2}{(\sqrt{2})^2} = 1$

(en ellips)

<u>Bild 1</u>:



<u>Bild 2</u>



(b) $x^2+6xy+y^2=3.$

$Q = x^2+6xy+y^2, \quad A=\begin{bmatrix}1 & 3\\ 3 & 1\end{bmatrix}, \ |A-\lambda E|=0 \Leftrightarrow$

$\begin{vmatrix}(1-\lambda) & 3\\ 3 & (1-\lambda)\end{vmatrix}=0 \Leftrightarrow \lambda^2-2\lambda-8=0 \Leftrightarrow \lambda_{1,2}=4,-2$

$\underline{\lambda_1=4}$: $\begin{bmatrix}-3 & 3 & | & 0\\ 3 & -3 & | & 0\end{bmatrix} \sim \begin{array}{ccc} x_1 & x_2 & \\ \hline \end{array}\begin{bmatrix}1 & -1 & | & 0\end{bmatrix} \sim$

$\begin{pmatrix}x_1\\x_2\end{pmatrix}=t\begin{pmatrix}1\\1\end{pmatrix}, \ t\neq 0 \Rightarrow \bar{p}_1=\frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}$

$\underline{\lambda_2=-2}$: $\begin{bmatrix}3 & 3 & | & 0\\ 3 & 3 & | & 0\end{bmatrix} \sim \begin{array}{ccc} x_1 & x_2 & \\ \hline \end{array}\begin{bmatrix}1 & 1 & | & 0\end{bmatrix} \sim$

$\begin{pmatrix}x_1\\x_2\end{pmatrix}=t\begin{pmatrix}-1\\1\end{pmatrix}, \ t\neq 0 \Rightarrow \bar{p}_2=\frac{1}{\sqrt{2}}\begin{pmatrix}-1\\1\end{pmatrix}$

Obs 1) $\bar{p}_1, \bar{p}_2$ är en ON bas bestående av egenvektorer till A

2) $A=PDP^t, \quad P=\frac{1}{\sqrt{2}}\begin{pmatrix}1 & -1\\ 1 & 1\end{pmatrix}, \ D=\begin{bmatrix}4 & 0\\ 0 & -2\end{bmatrix}$

3) Variabelbyte:

$\begin{pmatrix}x\\y\end{pmatrix}=P\begin{pmatrix}u\\v\end{pmatrix}=\frac{1}{\sqrt{2}}\begin{pmatrix}1 & -1\\ 1 & 1\end{pmatrix}\begin{pmatrix}u\\v\end{pmatrix} \quad \underline{0}$

$4u^2-2v^2=3 \Leftrightarrow \frac{u^2}{\left(\frac{\sqrt{3}}{2}\right)^2}-\frac{v^2}{\left(\sqrt{\frac{3}{2}}\right)^2}=1$

(en hyperbol)

Bild 1

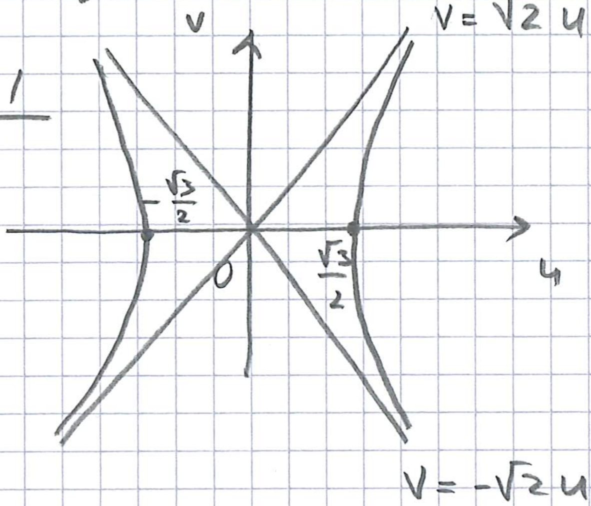
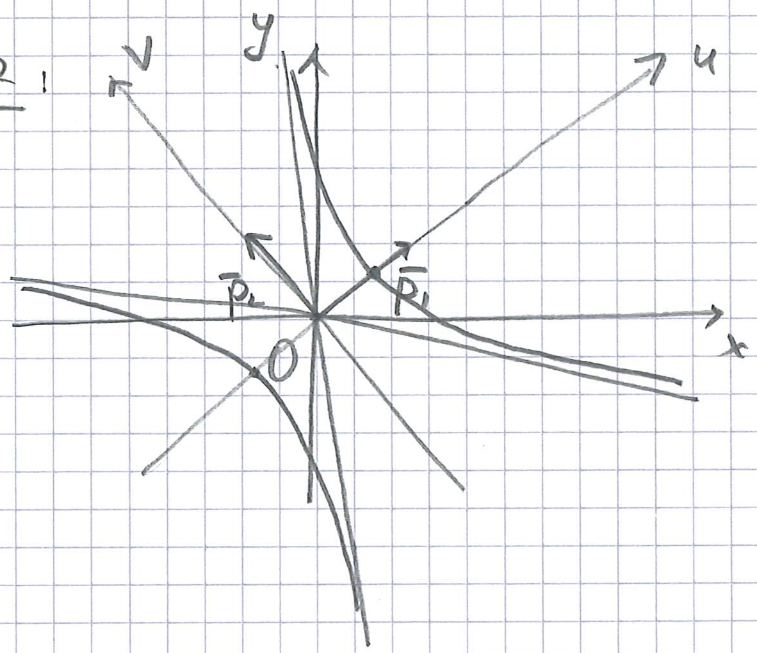


Bild 2:



8.24 Bestäm typen av de kvadratiska formerna.

(a) $Q = -3x^2 + 8xy + 3y^2$, $A = \begin{bmatrix} -3 & 4 \\ 4 & 3 \end{bmatrix}$,

$$|A - \lambda E| = 0 \iff \begin{vmatrix} (-3-\lambda) & 4 \\ 4 & (3-\lambda) \end{vmatrix} = 0$$

$\iff \lambda^2 - 25 = 0 \iff \lambda_{1,2} = \pm 5$ Obs $\lambda_1 \cdot \lambda_2 < 0$

$\Rightarrow$ Q är indefinit

(b) $Q = x^2 + 4xy + 4y^2$, $A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$,

$|A - \lambda E| = 0 \iff \begin{vmatrix} (1-\lambda) & 2 \\ 2 & (4-\lambda) \end{vmatrix} = 0 \iff$

$\lambda^2 - 5\lambda = 0 \iff \lambda_1 = 5, \ \lambda_2 = 0$

<u>Obs</u> $\min(\lambda_1, \lambda_2) = 0 \Rightarrow Q$ är positivt semidefinit

(c) $Q = x^2 + 8y^2 + 2z^2 + 8yz$, $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 8 & 4 \\ 0 & 4 & 2 \end{bmatrix}$

$|A - \lambda E| = 0 \iff \begin{vmatrix} (1-\lambda) & 0 & 0 \\ 0 & (8-\lambda) & 4 \\ 0 & 4 & (2-\lambda) \end{vmatrix} = 0 \iff$

$(1-\lambda)(\lambda^2 - 10\lambda) = 0 \iff \lambda_1 = 0, \ \lambda_2 = 1, \ \lambda_3 = 10$

<u>Obs</u> $\min(\lambda_1, \lambda_2, \lambda_3) = 0 \Rightarrow Q$ är positivt semidefinit.

(d) $Q = 3x^2 + 3y^2 + 6z^2 - 2xy$

$A = \begin{bmatrix} 3 & -1 & 0 \\ -1 & 3 & 0 \\ 0 & 0 & 6 \end{bmatrix}$, $|A - \lambda E| = 0 \iff \begin{vmatrix} (3-\lambda) & -1 & 0 \\ -1 & (3-\lambda) & 0 \\ 0 & 0 & (6-\lambda) \end{vmatrix} = 0$

$\iff (6-\lambda)(\lambda^2 - 6\lambda + 8) = 0 \iff \lambda_1 = 2, \ \lambda_2 = 4, \ \lambda_3 = 6$

<u>Obs</u> $\min(\lambda_1, \lambda_2, \lambda_3) > 0 \Rightarrow Q$ är positivt definit.

(c) $Q = x^2 + 4y^2 + 9z^2 - 4xy + 6xz - 12yz$

$$A = \begin{bmatrix} 1 & -2 & 3 \\ -2 & 4 & -6 \\ 3 & -6 & 9 \end{bmatrix}, \quad |A - \lambda E| = 0 \Leftrightarrow \begin{vmatrix} (1-\lambda) & -2 & 3 \\ -2 & (4-\lambda) & -6 \\ 3 & -6 & (9-\lambda) \end{vmatrix} = 0$$

$$(1-\lambda)\begin{vmatrix} (4-\lambda) & -6 \\ -6 & (9-\lambda) \end{vmatrix} + 2\begin{vmatrix} -2 & -6 \\ 3 & (9-\lambda) \end{vmatrix} + 3\begin{vmatrix} -2 & (4-\lambda) \\ 3 & -6 \end{vmatrix} = 0$$

$$(1-\lambda)(\lambda^2 - 13\lambda) + 2(2\lambda) + 3(3\lambda) = 0$$

$$\lambda\left((1-\lambda)(\lambda - 13) + 13\right) = 0 \Leftrightarrow$$

$$\lambda(14\lambda - \lambda^2) = 0 \Leftrightarrow \lambda_1 = 0 \qquad \lambda_2 = 14$$

(dubbelrot)

$\Rightarrow$ Q är positivt semidefinit.

<u>8.26</u>. (a) Bestäm max o min av

$Q = 2x^2 + y^2 + 3xy$ på cirkeln $x^2 + y^2 = 4 = 2^2$

$$A = \begin{bmatrix} 2 & 3/2 \\ 3/2 & 1 \end{bmatrix}, \quad |A - \lambda E| = 0 \Leftrightarrow \begin{vmatrix} (2-\lambda) & 3/2 \\ 3/2 & (1-\lambda) \end{vmatrix} = 0 \Leftrightarrow$$

$$\lambda^2 - 3\lambda - \frac{1}{4} = 0 \sim \lambda_{1,2} = \frac{3 \pm \sqrt{10}}{2}$$

$\Rightarrow$ $\max Q = 6 + 2\sqrt{10}$

$\min Q = 6 - 2\sqrt{10}$