(6) $Q = x^2 + 8y^2 + 2z^2 + 8yz$

på sfären $x^2 + y^2 + z^2 = 2$

$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 8 & 4 \\ 0 & 4 & 2 \end{bmatrix}$, $\lambda_{1,2,3} = 0, 1, 10$

$\Rightarrow$ max $Q = 10 \cdot 2 = 20$

min $Q = 2 \cdot 0 = 0$

Diff. ekvssystem.

① Bestäm alla lösningar till

$(*) \begin{cases} x_1' = 5x_2 \\ x_2' = 2x_1 \end{cases}$ o den lösning som satisfierar

villkoren: $x_1(0) = 1$, $x_2(0) = -1$

Obs $(*) \Leftrightarrow X' = A \cdot X$, där $A = \begin{bmatrix} 0 & 5 \\ 2 & 0 \end{bmatrix}$

$|A - \lambda E| = 0 \Leftrightarrow \begin{vmatrix} -\lambda & 5 \\ 2 & -\lambda \end{vmatrix} = 0 \Leftrightarrow \lambda^2 - 10 = 0 \Leftrightarrow \lambda_{1,2} = \pm\sqrt{10}$

$\lambda_1 = \sqrt{10}$:

$\left[\begin{array}{cc|c} -\sqrt{10} & 5 & 0 \\ 2 & -\sqrt{10} & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} -\sqrt{10} & 5 & 0 \end{array}\right] \sim$

$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = t \cdot \begin{pmatrix} 5 \\ \sqrt{10} \end{pmatrix}$, $t \neq 0$ $\Rightarrow$ $\bar{p}_1 = \begin{pmatrix} 5 \\ \sqrt{10} \end{pmatrix}$

$\lambda_2 = -\sqrt{10}$:

$\left[\begin{array}{cc|c} \sqrt{10} & 5 & 0 \\ 2 & \sqrt{10} & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} \sqrt{10} & 5 & 0 \end{array}\right] \sim$

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = t\begin{pmatrix} -5 \\ \sqrt{10} \end{pmatrix}, t \neq 0 \quad \Rightarrow \quad \bar{p}_2 = \begin{pmatrix} -5 \\ \sqrt{10} \end{pmatrix}$$

Den allmänna lsgen är $\bar{X}(t) = c_1 e^{\sqrt{10}t} \cdot \bar{p}_1 + c_2 e^{-\sqrt{10}\cdot t} \cdot \bar{p}_2$

$$= c_1 \cdot e^{\sqrt{10}t} \cdot \begin{pmatrix} 5 \\ \sqrt{10} \end{pmatrix} + c_2 e^{-\sqrt{10}t} \begin{pmatrix} -5 \\ \sqrt{10} \end{pmatrix}, \; c_1, c_2 \in R$$

Den partikulära lsgen:

$$\begin{cases} 1 = 5 \cdot c_1 - 5 \cdot c_2 \\ -1 = \sqrt{10} \cdot c_1 + \sqrt{10} \cdot c_2 \end{cases} \begin{vmatrix} \times \sqrt{10} \\ \times 5 \end{vmatrix} \oplus \Rightarrow \quad \sqrt{10} - 5 = 2 \cdot 5 \cdot \sqrt{10}\, c_1$$

$$\Rightarrow c_1 = \frac{\sqrt{10} - 5}{10\sqrt{10}} = \frac{1}{10} - \frac{1}{2\sqrt{10}}$$

$$c_2 = c_1 - \frac{1}{5} = \frac{1}{10} - \frac{1}{2\sqrt{10}} - \frac{1}{5} = -\frac{1}{10} - \frac{1}{2\sqrt{10}} \quad \Big| \Rightarrow$$

$$\bar{X}(t) = \left(\frac{1}{10} - \frac{1}{2\sqrt{10}}\right) \cdot e^{\sqrt{10}t} \begin{pmatrix} 5 \\ \sqrt{10} \end{pmatrix} + \left(-\frac{1}{10} - \frac{1}{2\sqrt{10}}\right) e^{-\sqrt{10}t} \cdot \begin{pmatrix} -5 \\ \sqrt{10} \end{pmatrix}$$

(2) $\begin{cases} x_1' = 3x_1 + 2x_2 \\ x_2' = 4x_1 + x_2 \end{cases}$ $\quad x_1(1) = 0, \; x_2(1) = 1$

$$X' = \underbrace{\begin{bmatrix} 3 & 2 \\ 4 & 1 \end{bmatrix}}_{= A} \cdot X, \quad |A - \lambda E| = 0 \iff \begin{vmatrix} (3-\lambda) & 2 \\ 4 & (1-\lambda) \end{vmatrix} = 0 \iff$$

$$\lambda^2 - 4\lambda - 5 = 0 \qquad \lambda_{1,2} = 5, -1.$$

$\lambda_1 = 5$: $\left[\begin{array}{cc|c} -2 & 2 & 0 \\ 4 & -4 & 0 \end{array}\right] \sim [\,1 \; -1 \,|\, 0\,] \sim \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = t\begin{pmatrix} 1 \\ 1 \end{pmatrix}, t \neq 0$

$$\bar{p}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$\underline{\lambda_2 = -1}$: $\left[\begin{array}{cc|c} 4 & 2 & 0 \\ 4 & 2 & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & \frac{1}{2} & 0 \end{array}\right] \sim \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = t\begin{pmatrix} -1/2 \\ 1 \end{pmatrix}, t \neq 0$

$\Rightarrow \bar{p}_2 = \begin{pmatrix} -1/2 \\ 1 \end{pmatrix}$

<u>Den allmänna lösgen:</u>

$\bar{x} = c_1 \cdot e^{5 \cdot t} \cdot \bar{p}_1 + c_2 \cdot e^{-t} \cdot \bar{p}_2 =$

$c_1 \cdot e^{5t}\begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 \cdot e^{-t}\begin{pmatrix} -1/2 \\ 1 \end{pmatrix}, \quad c_1, c_2 \in R.$

<u>Den partikulära lösgen:</u>

$\begin{cases} 0 = c_1 \cdot e^{5} - \frac{c_2}{2} \cdot e^{-1} \\ 1 = c_1 \cdot e^{5} + c_2 \cdot e^{-1} \end{cases} \Rightarrow 1 = \frac{3}{2} c_2 \cdot e^{-1} \Rightarrow$

$c_2 = \frac{2}{3} e, \quad c_1 = \frac{c_2}{2} \cdot e^{-1} \cdot e^{-5} = \frac{2 \cdot e \cdot e^{-1} \cdot e^{-5}}{3 \cdot 2} = \frac{1}{3} e^{-5}$

$\Rightarrow \bar{x}(t) = \frac{1}{3} e^{-5} \cdot e^{5t}\begin{pmatrix} 1 \\ 1 \end{pmatrix} + \frac{2}{3} e \cdot e^{-t}\begin{pmatrix} -1/2 \\ 1 \end{pmatrix}$

(3) Bestäm alla lösgar till

$\begin{cases} x_1' = x_2 + 2x_3 \\ x_2' = x_1 + x_3 \\ x_3' = 2x_1 + 2x_2 \end{cases} \qquad X' = \underbrace{\begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 1 \\ 2 & 2 & 0 \end{bmatrix}}_{= A} \cdot X$

$$|A-\lambda E|=0 \iff \begin{vmatrix} -\lambda & 1 & 2 \\ 1 & -\lambda & 1 \\ 2 & 2 & -\lambda \end{vmatrix} = 0 \iff$$

$$\begin{vmatrix} (3-\lambda) & (3-\lambda) & (3-\lambda) \\ 1 & -\lambda & 1 \\ 2 & 2 & -\lambda \end{vmatrix} = 0 \quad \text{eller} \quad (3-\lambda)\begin{vmatrix} 1 & 1 & 1 \\ 1 & -\lambda & 1 \\ 2 & 2 & -\lambda \end{vmatrix} = 0$$

$$\iff (3-\lambda)\begin{vmatrix} 1 & 0 & 0 \\ 1 & (-\lambda-1) & 0 \\ 2 & 0 & (-\lambda-2) \end{vmatrix} = 0 \iff \lambda_1 = 3,\ \lambda_2 = -1,\ \lambda_3 = -2$$

$\underline{\lambda_1 = 3}$:

$$\left[\begin{array}{ccc|c} -3 & 1 & 2 & 0 \\ 1 & -3 & 1 & 0 \\ 2 & 2 & -3 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} \cancel{0} & \cancel{0} & \cancel{0} & \cancel{0} \\ 1 & -3 & 1 & 0 \\ 2 & 2 & -3 & 0 \end{array}\right] \sim$$

$$\left[\begin{array}{ccc|c} 1 & -3 & 1 & 0 \\ 2 & 2 & -3 & 0 \end{array}\right]^{\times(-2)} \sim \left[\begin{array}{ccc|c} 1 & -3 & 1 & 0 \\ 0 & 8 & -5 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & -3 & 1 & 0 \\ 0 & 1 & -\frac{5}{8} & 0 \end{array}\right] \sim$$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = t\begin{pmatrix} 7/8 \\ 5/8 \\ 1 \end{pmatrix}, \quad t \neq 0 \quad \Rightarrow \quad \bar{p}_1 = \begin{pmatrix} 7/8 \\ 5/8 \\ 1 \end{pmatrix}$$

$\underline{\lambda_2 = -1}$:

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 0 \\ 1 & 1 & 1 & 0 \\ 2 & 2 & 1 & 0 \end{array}\right]^{\times -2} \sim \left[\begin{array}{ccc|c} 1 & 1 & 2 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & -3 & 0 \end{array}\right] \sim$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right] \sim \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = t\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} \Rightarrow \bar{p}_2 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$$

$\underline{\lambda_3 = -2}$:

$$\left[\begin{array}{ccc|c} 2 & 1 & 2 & 0 \\ 1 & 2 & 1 & 0 \\ 2 & 2 & 2 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 2 & 1 & 2 & 0 \\ 0 & 1 & 0 & 0 \end{array}\right] \begin{matrix} -2 \\ \leftarrow \end{matrix} \sim$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 0 & -3 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{array}\right] \sim \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = t \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

$$\Rightarrow \bar{p}_3 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

Den allmänna lösgen:

$$\bar{x}(t) = c_1 \cdot e^{3t} \cdot \begin{pmatrix} 7/8 \\ 5/8 \\ 1 \end{pmatrix} + c_2 \cdot e^{-t} \cdot \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + c_3 \cdot e^{-2t} \cdot \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

$c_1, c_2, c_3 \in R$

(4)
$$\begin{cases} x_1' = x_1 + x_2 + x_3 \\ x_2' = -x_3 \\ x_3' = -x_2 \end{cases}$$

$\underline{0}$ $x_1(0) = 5$, $x_2(0) = -4$, $x_3(0) = 0$

$$X' = \underbrace{\begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{bmatrix}}_{A} \cdot X, \quad |A - \lambda E| = 0 \Leftrightarrow \begin{vmatrix} (1-\lambda) & 1 & 1 \\ 0 & -\lambda & -1 \\ 0 & -1 & -\lambda \end{vmatrix} = 0$$

$\Leftrightarrow (1-\lambda)(\lambda^2 - 1) = 0 \Leftrightarrow \lambda_1 = 1$ (dubbelrot)

$\lambda_2 = -1$ (enkelrot).

$\underline{\lambda_1 = 1:}$

$$\left[\begin{array}{ccc|c} 0 & 1 & 1 & 0 \\ 0 & -1 & -1 & 0 \\ 0 & -1 & -1 & 0 \end{array}\right] \sim [0 \; 1 \; 1 \,|\, 0] \sim$$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} s \\ -t \\ t \end{pmatrix} = s \underbrace{\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}}_{\bar{p}_1} + t \underbrace{\begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}}_{\bar{p}_2},$$ s, t är inte nollor samtidigt.

$\underline{\lambda_2 = -1}$:

$$\left[\begin{array}{ccc|c} 2 & 1 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ \cancel{0} & \cancel{-1} & \cancel{1} & \cancel{0} \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & \frac{1}{2} & 0 \\ 0 & 1 & -1 & 0 \end{array}\right] \sim$$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = t \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}, \; t \neq 0 \Rightarrow \bar{p}_3 = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$$

Den allmänna lösningen:

$$\bar{x}(t) = c_1 \cdot e^t \cdot \bar{p}_1 + c_2 e^t \bar{p}_2 + c_3 \cdot e^{-t} \cdot \bar{p}_3 =$$

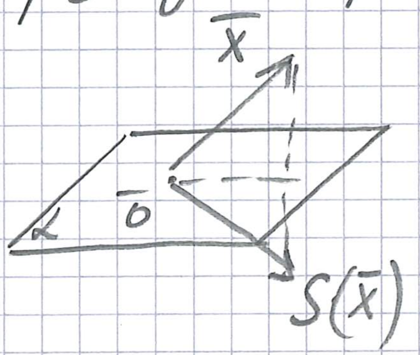
$$= c_1 e^t \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + c_2 \cdot e^t \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} + c_3 e^{-t} \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}, \; c_1, c_2, c_3 \in R$$

Den partikulära lösningen:

$$\begin{cases} 5 = c_1 - c_3 \\ -4 = -c_2 + c_3 \\ 0 = c_2 + c_3 \end{cases} \Rightarrow c_3 = -2, \; c_2 = 2, \; c_1 = 3 \Rightarrow$$

$$\bar{x}(t) = 3 \cdot e^t \bar{p}_1 + 2e^t \bar{p}_2 - 2e^{-t} \bar{p}_3$$

(5) Bestäm alla lsgar till ekv $X' = AX$ där A är avbildningsmatrisen för spegling i planet $2x_1 + 2x_2 + x_3 = 0$ (α):



Obs A har två egenvärden

$\lambda_1 = 1$ o $\lambda_2 = -1$

$\lambda_1 = 1$:
$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = t \overbrace{\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}}^{=\bar{p}_1} + s \overbrace{\begin{pmatrix} 0 \\ 1 \\ -2 \end{pmatrix}}^{=\bar{p}_2},$$
t, s är inte nollor samtidigt.

$\lambda_2 = -1$:
$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = t \cdot \overbrace{\begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}}^{=\bar{p}_3}, \quad t \neq 0$$

Den allmänna lsgen är

$$\bar{X}(t) = c_1 \cdot e^{t} \cdot \bar{p}_1 + c_2 \cdot e^{t} \cdot \bar{p}_2 + c_3 \cdot e^{-t} \bar{p}_3 .$$

$c_1, c_2, c_3 \in \mathbb{R}$.