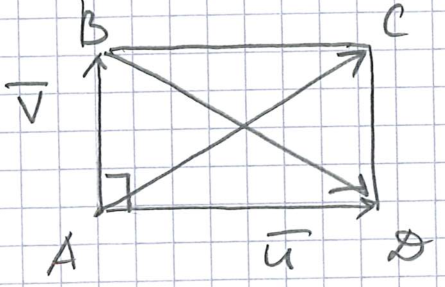
1.30 Anta att vektorerna $\bar{u}$ o $\bar{v}$ är vinkelräta.

- Visa att $|\bar{u}+\bar{v}| = |\bar{u}-\bar{v}|$ $(*)$




Obs 1) $\bar{u}+\bar{v} = \overline{AC}$ o

$\bar{u}-\bar{v} = \overline{BD}$

2) ABCD är en rektangel,

Pytagoras sats $\Rightarrow |\overline{AC}| = |\overline{BD}|$

- Antag att $\bar{u} = \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix}$ o $\bar{v} = \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix}$ (i ett ortonormerat system)

Visa att om $\bar{u} \perp \bar{v}$ så är

$x_1 \cdot x_2 + y_1 \cdot y_2 + z_1 z_2 = 0$.

Obs 1) För varje vektor $\bar{u} = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}$

i ett ortonormerat system har vi

$|\bar{u}| = \sqrt{u_1^2 + u_2^2 + u_3^2}$ (Pytagoras sats). $(**)$

2) $\bar{u}+\bar{v} = \begin{pmatrix} x_1+x_2 \\ y_1+y_2 \\ z_1+z_2 \end{pmatrix}$ o $\bar{u}-\bar{v} = \begin{pmatrix} x_1-x_2 \\ y_1-y_2 \\ z_1-z_2 \end{pmatrix}$

$(*)$ o $(**)$ $\Rightarrow (x_1+x_2)^2 + (y_1+y_2)^2 + (z_1+z_2)^2 =$

$= (x_1-x_2)^2 + (y_1-y_2)^2 + (z_1-z_2)^2$. Förkorta,

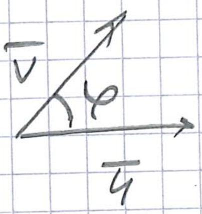
$2x_1 x_2 + 2y_1 y_2 + 2z_1 z_2 = -2x_1 x_2 - 2y_1 y_2 - 2z_1 z_2$ eller

$x_1x_2 + y_1y_2 + z_1z_2 = 0$ VhV.

<u>1.35</u> $\bar{u} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \quad \bar{v} = \begin{pmatrix} -1 \\ 3 \\ 0 \end{pmatrix}$

(a) Bestäm $\bar{u}\cdot\bar{v} = 1\cdot(-1) + 2\cdot 3 + 3\cdot 0 = 5$

(b) Bestäm vinkeln mellan $\bar{u}$ o $\bar{v}$.




<u>Obs</u> 1) $\bar{u}\cdot\bar{v} = |\bar{u}|\cdot|\bar{v}|\cdot\cos\varphi$

2) $|\bar{u}| = \sqrt{1^2+2^2+3^2} = \sqrt{14}$

$|\bar{v}| = \sqrt{(-1)^2+3^2+0^2} = \sqrt{10}$

$$\Rightarrow \quad \cos\varphi = \frac{\bar{u}\cdot\bar{v}}{|\bar{u}|\cdot|\bar{v}|} = \frac{5}{\sqrt{14}\cdot\sqrt{10}}$$

$$\Rightarrow \quad \varphi = \arccos\frac{5}{\sqrt{140}}$$

<u>1.38</u> $\bar{u} = \begin{pmatrix} 4 \\ 7 \end{pmatrix}$

(a) Bestäm en vektor som är vinkelrät mot $\bar{u}$.

Sätt $\bar{v} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$. Anta att $\bar{v} \perp \bar{u}$. Så är $\bar{v}\cdot\bar{u} = 0$

<u>Obs</u> $\bar{v}\cdot\bar{u} = 4x_1 + 7x_2 = 0$

Välj (till ex.) $x_1 = -7, \; x_2 = 4$

(b) Bestäm alla <u>enhetsvektorer</u> som är vinkelräta mot u.

Repetera att enhetsvektor har längd 1.

Vi har två villkor:

$$\begin{cases} 4x_1 + 7x_2 = 0 & (1) \\ x_1^2 + x_2^2 = 1 & (2) \; (|\bar{v}| = 1) \end{cases}$$

$(1) \Rightarrow x_2 = -\frac{4}{7}x_1$. Sätt in i (2):

$x_1^2 + \left(-\frac{4}{7}x_1\right)^2 = 1$ eller $\left(1 + \frac{16}{49}\right)x_1^2 = 1$

eller $x_1^2 = \frac{49}{65} \iff x_1 = \pm\frac{7}{\sqrt{65}}$

$\Rightarrow \bar{v}_1 = \left(\frac{7}{\sqrt{65}}, -\frac{4}{\sqrt{65}}\right)$, $\bar{v}_2 = \left(-\frac{7}{\sqrt{65}}, \frac{4}{\sqrt{65}}\right)$.

**1.39** Bestäm $a$ s.a. vektorerna $\underbrace{\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}}_{\bar{u}}$ & $\underbrace{\begin{pmatrix} a \\ -1 \\ 5 \end{pmatrix}}_{\bar{v}}$

blir vinkelräta

Obs $\bar{u} \cdot \bar{v} = 0 = a + 2(-1) + 3 \cdot 5 = 0$

$\Rightarrow a = -13$

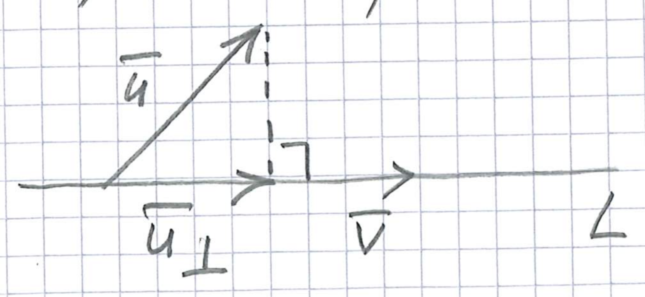
**1.41** Bestäm ortogonala projektionen $\bar{u}_L$

av vektorn $\bar{u}$ på linjen $L$ med riktningsvektor

$\bar{v}$.

(a) $\bar{u} = \begin{pmatrix} 3 \\ 0 \\ -1 \end{pmatrix}$ $\bar{v} = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$

Rita!



Repetera att $\bar{u}_\perp = \underbrace{\dfrac{\bar{u}\cdot\bar{v}}{|\bar{v}|^2}}_{\text{en Konstant.}} \cdot \bar{v}$

$\bar{u}\cdot\bar{v} = 3\cdot 1 + 0\cdot 2 + (-1)\cdot 2 = 1$

$|\bar{v}|^2 = 1^2 + 2^2 + 2^2 = 9$

$\Rightarrow \bar{u}_\perp = \frac{1}{9}\cdot \underset{\bar{v}}{\underset{\shortparallel}{(1,2,2)}}$ o $|\bar{u}_\perp| = |\frac{1}{9}\bar{v}| =$

$= \frac{1}{9}|\bar{v}| = \frac{\sqrt{9}}{9} = \frac{1}{3}$

(b) $\bar{u} = \begin{pmatrix} -2 \\ -4 \\ 3 \end{pmatrix}$ o $\bar{v} = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$

$\bar{u}\cdot\bar{v} = (-2)\cdot 1 + (-4)\cdot 2 + 3\cdot 0 = -10$

$|\bar{v}|^2 = 1^2 + 2^2 + 0^2 = 5$

$\Rightarrow \bar{u}_\perp = -\frac{10}{5}(1,2,0)$ o $|\bar{u}_\perp| = |-2\cdot\bar{v}| =$

$= 2\cdot\sqrt{5}$

1.42. Låt $\bar{u}, \bar{v}$ vara vektorer s.a.

$|\bar{u}| = |\bar{v}|$ o vinkeln dem emellan är $\frac{2\pi}{3}$

Bestäm vinkeln $\varphi$ mellan vektorerna $\bar{u} + 2\bar{v}$ o $3\bar{u} - \bar{v}$

Inför: $\bar{a} = \bar{u} + 2\bar{v}$, $\bar{b} = 3\bar{u} - \bar{v}$

Notera att $\cos\varphi = \dfrac{\bar{a}\cdot\bar{b}}{|\bar{a}|\cdot|\bar{b}|}$.

Sätt $k = |\bar{u}| = |\bar{v}|$

Räkna i

$\bar{a}\cdot\bar{b} = (\bar{u}+2\bar{v})(3\bar{u}-\bar{v}) = \Big/$ använd egenskaper hos skalärprodukten $\Big/ =$

$= 3\bar{u}\cdot\bar{u} + 6\,\bar{v}\cdot\bar{u} - \bar{u}\cdot\bar{v} - 2\bar{v}\cdot\bar{v} =$

$= 3|\bar{u}|^2 + 5\,\bar{u}\cdot\bar{v} - 2|\bar{v}|^2 = \Big/ \bar{u}\cdot\bar{v} = |\bar{u}||\bar{v}|\cdot\cos\frac{2\pi}{3} = -\frac{k^2}{2}$, $\cos\frac{2\pi}{3} = -\frac{1}{2}$ $\Big/$

$= 3k^2 - \frac{5}{2}k^2 - 2k^2 = -\frac{3}{2}k^2$

Analogt,

$|\bar{a}|^2 = \bar{a}\cdot\bar{a} = (\bar{u}+2\bar{v})(\bar{u}+2\bar{v}) = |\bar{u}|^2 + 4\bar{u}\cdot\bar{v} + 4|\bar{v}|^2 =$

$= k^2 - 2k^2 + 4k^2 = 3k^2$

$|\bar{b}|^2 = \bar{b}\cdot\bar{b} = (3\bar{u}-\bar{v})(3\bar{u}-\bar{v}) = 9|\bar{u}|^2 - 6\bar{u}\cdot\bar{v} + |\bar{v}|^2 =$

$= 9k^2 + 3k^2 + k^2 = 13k^2$

$\Rightarrow \cos\varphi = \dfrac{-\frac{3}{2}k^2}{k\cdot\sqrt{3}\cdot k\sqrt{13}} = \dfrac{-\frac{3}{2}}{\sqrt{39}}$

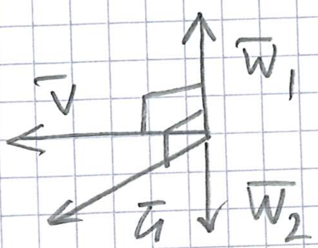
$\Rightarrow \varphi = \arccos\left(\dfrac{-3/2}{\sqrt{39}}\right).$

1.52 Bestäm alla enhetsvektorer som är vinkelräta mot $\bar{u} = \begin{pmatrix}1\\2\\3\end{pmatrix}$ o $\bar{v} = \begin{pmatrix}-2\\0\\3\end{pmatrix}$.

Vi söker vektorer $\overline{w}$ s.a.

1) $|\overline{w}| = 1$ o 2) $\overline{w} \perp \overline{u}, \overline{v}$

Obs (i) det finns bara två sådana vektorer




. Dessutom $\overline{w}_2 = -\overline{w}_1$

(ii) $\overline{u} \times \overline{v} \perp \overline{u}, \overline{v}$

Finn $\overline{u} \times \overline{v} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \times \begin{pmatrix} -2 \\ 0 \\ 3 \end{pmatrix} = \begin{pmatrix} 6 \\ -9 \\ 4 \end{pmatrix}$

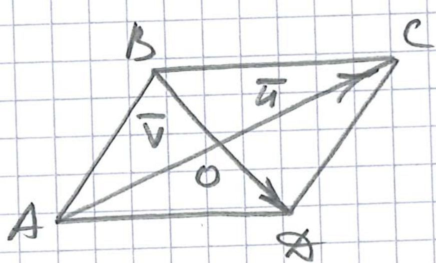
Sätt $\overline{w}_1 = \dfrac{1}{|\overline{u} \times \overline{v}|} \cdot \overline{u} \times \overline{v}$. Notera att

$|\overline{w}_1| = 1$ o $\overline{w}_1 \perp \overline{u}, \overline{v}$.

$|\overline{u} \times \overline{v}| = \sqrt{6^2 + (-9)^2 + 4^2} = \sqrt{133}$

$\Rightarrow \overline{w}_1 = \dfrac{1}{\sqrt{133}} \cdot (6, -9, 4)$ o $\overline{w}_2 = -\dfrac{1}{\sqrt{133}} (6, -9, 4)$

1.55

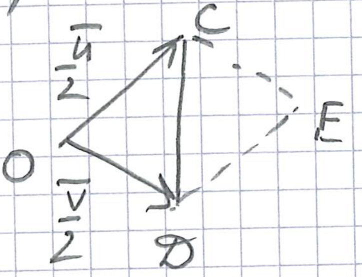



$\overline{u} = \begin{pmatrix} 3 \\ 1 \\ -2 \end{pmatrix}$, $\overline{v} = \begin{pmatrix} 1 \\ -3 \\ 4 \end{pmatrix}$

Bestäm arean $A$ av parallellogramen ABCD.

Obs i) $A = 4 \cdot$ arean av $\triangle OCD$.

2) arean av $\Delta\, OCD = \frac{1}{2} \cdot \left| \frac{\bar{u}}{2} \times \frac{\bar{v}}{2} \right| = \frac{1}{8} |\bar{u} \times \bar{v}|$




ty arean av parallellogrammen $OCED = \left| \frac{\bar{u}}{2} \times \frac{\bar{v}}{2} \right|$

Räkna: $\bar{u} \times \bar{v} = \begin{pmatrix} 3 \\ 1 \\ -2 \end{pmatrix} \times \begin{pmatrix} 1 \\ -3 \\ 4 \end{pmatrix} = \begin{pmatrix} -2 \\ -14 \\ -10 \end{pmatrix} = -2 \begin{pmatrix} 1 \\ 7 \\ 5 \end{pmatrix}$

o $|\bar{u} \times \bar{v}| = 2 \cdot \sqrt{1^2 + 7^2 + 5^2} = 2 \cdot \sqrt{75} = 10\sqrt{3}$

$\Rightarrow A = 4 \cdot \frac{1}{8} \cdot 10\sqrt{3} = 5\sqrt{3}$

**1.56** Bestäm tre enhetsvektorer $\bar{u}, \bar{v}, \bar{w}$ som är parvis ortogonala o dessutom

(i) $\bar{u}$ // linjen som går genom $\underset{A}{(1,0,2)}$ o $\underset{B}{(2,1,3)}$

(ii) $\bar{v} \perp \bar{a} = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$.

Vi söker $\bar{u}, \bar{v}, \bar{w}$ s.a.

$$\begin{cases} 1) & |\bar{u}| = |\bar{v}| = |\bar{w}| = 1 \\ 2) & \bar{u} \perp \bar{v},\ \bar{u} \perp \bar{w},\ \bar{v} \perp \bar{w} \\ 3) & \bar{u} \parallel \overline{AB} \\ 4) & \bar{v} \perp \bar{a} \end{cases}$$

Start: $\overline{AB} = B - A = (2-1, 1-0, 3-2) = (1,1,1)$

o $|\overline{AB}| = \sqrt{3}$

$(3) \Rightarrow \bar{u} = \pm \frac{1}{\sqrt{3}}(1,1,1)$

Notera att $\bar{v} \perp \bar{a}$ o $\bar{v} \perp \bar{u}$

$$\Rightarrow \bar{v} = \pm \frac{1}{|\bar{a} \times \bar{u}|} \cdot \bar{a} \times \bar{u}$$

$$\bar{a} \times \bar{u} = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} \times \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{3}} \begin{pmatrix} 2 \\ -2 \\ 0 \end{pmatrix} = \frac{2}{\sqrt{3}} \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$$

$$|\bar{a} \times \bar{u}| = \frac{2}{\sqrt{3}} \cdot \sqrt{1^2 + (-1)^2} = 2\sqrt{\frac{2}{3}}$$

$$\Rightarrow \bar{v} = \pm \sqrt{\frac{3}{8}} \cdot \frac{2}{\sqrt{3}} \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} = \pm \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$$

$$\bar{w} = \pm \bar{u} \times \bar{v} = \pm \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \times \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} =$$

$$= \pm \frac{1}{\sqrt{6}} \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}$$

**1.58** (a) Visa att om $\bar{u} + \bar{v} + \bar{w} = \bar{0}$ så är

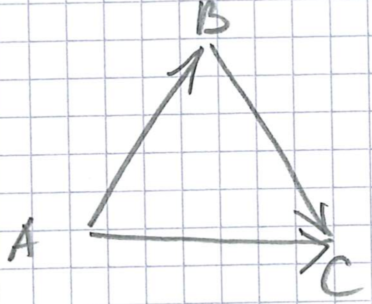
$$\bar{u} \times \bar{v} = \bar{v} \times \bar{w}$$

Obs $\bar{w} = -\bar{u} - \bar{v}$, Så är $H.L. = \bar{v} \times \bar{w} =$

$$= \bar{v} \times (-\bar{u} - \bar{v}) = \bar{v} \times (-\bar{u}) + \underbrace{\bar{v} \times (-\bar{v})}_{= \bar{0} \text{ ty } \bar{v} \parallel -\bar{v}} =$$

$$= -\bar{v} \times \bar{u} = \bar{u} \times \bar{v} = V.L.$$

(6) Visa sinussatsen:



$$\underbrace{\frac{\sin C}{|\overline{AB}|}=\frac{\sin B}{|\overline{AC}|}}_{(*)}\overbrace{=\frac{\sin A}{|\overline{BC}|}}^{(**)}$$

Låt visa $(*)$:

<u>Obs</u> Arean av $\triangle ABC = |\overline{AC}|\cdot|\overline{BC}|\cdot\sin C =$

$= |\overline{AB}|\cdot|\overline{BC}|\cdot\sin B$

(geometrisk tolkning av kryssprodukten)

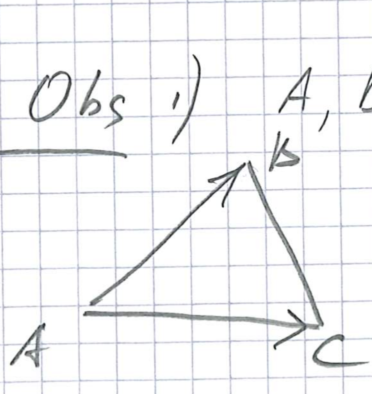
Förkorta med $|\overline{BC}| \Rightarrow |\overline{AC}|\cdot\sin C = |\overline{AB}|\cdot\sin B$

eller $\dfrac{\sin C}{|\overline{AB}|} = \dfrac{\sin B}{|\overline{AC}|}$.

Analogt med $(**)$. Prova själva!

<u>1.67.</u> Undersök om punkterna $\overset{=A}{(1,1,1)}$, $\overset{=B}{(0,2,1)}$, $\underset{=C}{(-1,0,1)}$ o $\underset{=D}{(2,2,-3)}$ ligger i samma plan.

<u>Obs</u> 1) A, B, C ligger i samma plan



2) A, B, C, D ligger i samma plan $\iff \overline{AB}\times\overline{AC} \perp \overline{AD}$.

Räkna: $\overline{AB} = B - A = (0,2,1) - (1,1,1) = (-1,1,0)$

$\overline{AC} = C - A = (-1,0,1) - (1,1,1) = (-2,-1,0)$

$\overline{AD} = D - A = (2,2,-3) - (1,1,1) = (1,1,-4)$

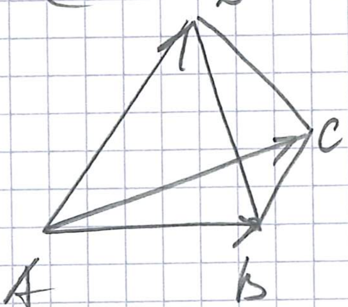
$$\overline{AB} \times \overline{AC} = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} \times \begin{pmatrix} -2 \\ -1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 3 \end{pmatrix}$$

Kontrollera $(\overline{AB} \times \overline{AC}) \cdot \overline{AD} =$

$= 0 \cdot 1 + 0 \cdot 1 + (3) \cdot (-4) = -12 \neq 0 \Rightarrow$

A, B, C, D ligger ej i samma plan.

1.68 Bestäm volymen av tetraedern som har hörn i punkterna $\overset{A}{(1,0,0)}$, $\overset{B}{(0,2,0)}$, $\underset{C}{(0,0,3)}$ $\underset{D}{o}$ $\underset{D}{(6,-12,12)}$




$V = \frac{1}{3}$ basen $\times$ höjden.

Obs 1) basen $= \frac{1}{2} |\overline{AB} \times \overline{AC}|$.

$$\bar{n} = \overline{AB} \times \overline{AC} = \begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix} \times \begin{pmatrix} -1 \\ 0 \\ 3 \end{pmatrix} = \begin{pmatrix} 6 \\ 3 \\ 2 \end{pmatrix} \quad o$$

$|\bar{n}| = \sqrt{6^2 + 3^2 + 2^2} = \sqrt{36 + 9 + 4} = 7 \Rightarrow$ basen $= \frac{1}{2} \cdot 7$

2) höjden $= |\text{proj}_{\bar{n}} \overline{AD}|$

$$\text{proj}_{\bar{n}} \overline{AD} = \frac{\overline{AD} \cdot \bar{n}}{|\bar{n}|^2} \cdot \bar{n} = \frac{(5,-12,12)\cdot(6,3,2)}{49} \cdot \bar{n} =$$

$$= \frac{5\cdot 6 + (-12)\cdot 3 + 12\cdot 2}{49} \cdot \bar{n} = \frac{18}{49} \cdot \bar{n}$$

$$\Rightarrow \text{höjda} = \left| \frac{18}{49} \cdot \bar{n} \right| = \frac{18}{7}$$

$$\Rightarrow V = \frac{1}{3} \cdot \frac{7}{2} \cdot \frac{18}{7} = 3$$