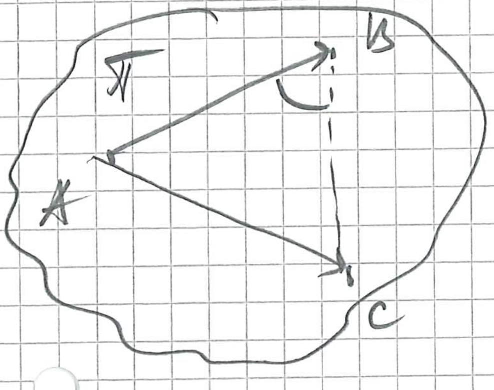
① (i) plans ekv genom punkterna $A(1,2,3)$ $B(2,3,4)$, $C(0,2,5)$



$\overline{AB} = B - A = (2,3,4) - (1,2,3) = (1,1,1)$

$\overline{AC} = C - A = (0,2,5) - (1,2,3) = (-1,0,2)$

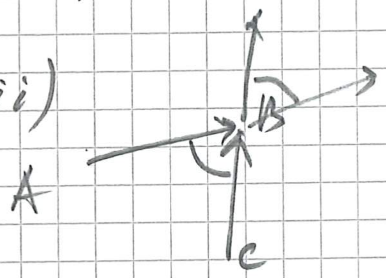
$$\overline{n} = \overline{AB} \times \overline{AC} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \times \begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix} = \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix}$$

$\pi: \; 2x - 3y + z + D = 0$

Insättning C i ekv: $2\cdot 0 \overbrace{- 3\cdot 2 + 5}^{-1} + D = 0$

$\Rightarrow D = 1 \Rightarrow \pi: \; \underline{2x - 3y + z + 1 = 0}$

(ii)



$\overline{CB} = B - C = (2,3,4) - (0,2,5) =$

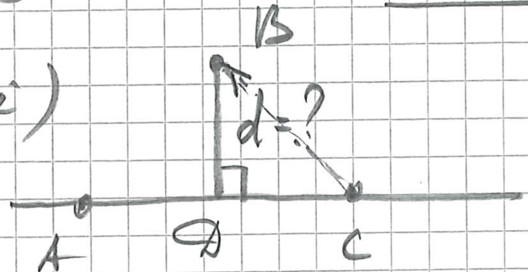
$= (2,1,-1), \quad |\overline{CB}| = \sqrt{2^2 + 1^2 + (-1)^2} = \sqrt{6}$

$|\overline{AB}| = \sqrt{1^2 + 1^2 + 1^2} = \sqrt{3}$

$$\cos B = \frac{\overline{AB} \cdot \overline{CB}}{|\overline{AB}| \cdot |\overline{CB}|} = \frac{2\cdot 1 + \overbrace{1\cdot 1 + (-1)\cdot 1}^{0}}{\sqrt{6}\,\sqrt{3}} = \frac{2}{3\sqrt{2}} =$$

$= \dfrac{\sqrt{2}}{3} \Rightarrow \underline{B = \arccos \dfrac{\sqrt{2}}{3}}$

(iii)



Obs 1) $\underset{\overline{AC}}{\text{proj}} \overset{\overset{\displaystyle \overline{CD}}{\shortparallel}}{\overline{CB}} = \dfrac{\overline{AC} \cdot \overline{CB}}{|\overline{AC}|^2} \cdot \overline{AC}$

2) $\left| \underset{\overline{AC}}{\text{proj}} \overline{CB} \right| = \dfrac{|\overline{AC} \cdot \overline{CB}|}{|\overline{AC}|} = \dfrac{|(-1)\cdot 2 + 0\cdot 1 + 2\cdot(-1)|}{\sqrt{(-1)^2 + 2^2}} = \dfrac{4}{\sqrt{5}}$

Пифагор:

$$d = \sqrt{BC^2 - DC^2} = \sqrt{6 - \frac{16}{5}} = \sqrt{\frac{30-16}{5}} =$$

$$= \sqrt{\frac{14}{5}};$$

2 $T = \begin{pmatrix} 1 & 0 & -2 \\ -3 & 1 & 4 \\ 2 & -3 & 4 \end{pmatrix}$

(i) $T^{-1}: \left[\begin{array}{ccc|ccc} 1 & 0 & -2 & 1 & 0 & 0 \\ -3 & 1 & 4 & 0 & 1 & 0 \\ 2 & -3 & 4 & 0 & 0 & 1 \end{array}\right] {}^{\times 3, -2} \sim \left[\begin{array}{ccc|ccc} 1 & 0 & -2 & 1 & 0 & 0 \\ 0 & 1 & -2 & 3 & 1 & 0 \\ 0 & -3 & 8 & -2 & 0 & 1 \end{array}\right] \times 3$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & -2 & 1 & 0 & 0 \\ 0 & 1 & -2 & 3 & 1 & 0 \\ 0 & 0 & 2 & 7 & 3 & 1 \end{array}\right] \times \frac{1}{2} \sim \left[\begin{array}{ccc|ccc} 1 & 0 & -2 & 1 & 0 & 0 \\ 0 & 1 & -2 & 3 & 1 & 0 \\ 0 & 0 & 1 & 7/2 & 3/2 & 1/2 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 8 & 3 & 1 \\ 0 & 1 & 0 & 10 & 4 & 1 \\ 0 & 0 & 1 & 7/2 & 3/2 & 1/2 \end{array}\right]$$

$$\underbrace{}_{\parallel} \; T^{-1} = S$$

(ii) $$\begin{bmatrix} 1 & 0 & -2 \\ -3 & 1 & 4 \\ 2 & -3 & 4 \end{bmatrix} \cdot \begin{bmatrix} 8 & 3 & 1 \\ 10 & 4 & 1 \\ 7/2 & 3/2 & 1/2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

ok!

(iii) $\bar{x} = F \cdot \begin{pmatrix} 4 \\ 5 \\ 2 \end{pmatrix} = G \cdot X$

$\begin{pmatrix} 4 \\ 5 \\ 2 \end{pmatrix} = Y$, $\quad X \overset{?}{=}$

$\bar{x} = G \cdot S \cdot Y \Rightarrow X = S \cdot Y$

Så är $X = \begin{bmatrix} 8 & 3 & 1 \\ 10 & 4 & 1 \\ 7/2 & 3/2 & 1/2 \end{bmatrix} \begin{bmatrix} 4 \\ 5 \\ 2 \end{bmatrix} = \begin{bmatrix} \overbrace{32+15}^{47}+2 \\ 40+20+2 \\ \underbrace{14+\frac{15}{2}+1}_{} \end{bmatrix} =$

49

$15 \cdot 3/2 = \frac{45}{2}$

$= \begin{bmatrix} 49 \\ 62 \\ 45/2 \end{bmatrix}$

③ (*) $A\bar{x} = \bar{b}$, $\quad A = \begin{bmatrix} 1 & -2 \\ 3 & -1 \\ 1 & 1 \end{bmatrix}$, $\bar{b} = \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}$

(i) Gauss elim.

$\left[\begin{array}{cc|c} 1 & -2 & 2 \\ 3 & -1 & 3 \\ 1 & 1 & 1 \end{array}\right] \overset{\times(-3),\,-1}{\sim} \left[\begin{array}{cc|c} 1 & -2 & 2 \\ 0 & 5 & -3 \\ 0 & 3 & -1 \end{array}\right] \begin{matrix} \\ \times\frac{1}{5} \\ \times\frac{1}{3} \end{matrix} \sim \left[\begin{array}{cc|c} 1 & -2 & 2 \\ 0 & 1 & -3/5 \\ 0 & 1 & -1/3 \end{array}\right] \#$

$\Rightarrow \emptyset \Rightarrow$ (*) är överbestämt.

(ii) normalekv: $A^T A \bar{x} = A^T \bar{b}$ (**)

$A^t A = \begin{bmatrix} 1 & 3 & 1 \\ -2 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & -2 \\ 3 & -1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} (1+9+1) & (-2-3+1) \\ (-2-3+1) & (4+1+1) \end{bmatrix} =$

4

$$= \begin{bmatrix} 11 & -4 \\ -4 & 6 \end{bmatrix}, \quad A^t \bar{b} = \begin{bmatrix} 1 & 3 & 1 \\ -2 & -1 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 2+9+1 \\ -4-3+1 \end{bmatrix} =$$

$$= \begin{bmatrix} 12 \\ -6 \end{bmatrix}$$

<u>Lös (**):</u>

$$\left[\begin{array}{cc|c} 11 & -4 & 12 \\ -4 & 6 & -6 \end{array}\right] \times 2 \sim \left[\begin{array}{cc|c} 3 & 8 & 0 \\ -4 & 6 & -6 \end{array}\right] \times \tfrac{1}{2} \sim$$

$$\sim \left[\begin{array}{cc|c} 3 & 8 & 0 \\ -2 & 3 & -3 \end{array}\right] \times 1 \sim \left[\begin{array}{cc|c} 3 & 8 & 0 \\ 1 & 11 & -3 \end{array}\right] \times (-3) \sim$$

$$\begin{array}{c} \begin{array}{cc} x_1 & x_2 \end{array} \\ \left[\begin{array}{cc|c} 0 & -25 & 9 \\ 1 & 11 & -3 \end{array}\right] \end{array} \Rightarrow \quad -25x_2 = 9 \Rightarrow x_2 = -9/25$$

$$x_1 + 11x_2 = -3 \Rightarrow x_1 = -3 + \frac{11 \cdot 9}{25} =$$

$$= \frac{-75+99}{25} = \frac{24}{25} \Rightarrow \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \frac{1}{25}\begin{pmatrix} 24 \\ -9 \end{pmatrix} =$$

$$= \underline{\frac{3}{25} \cdot \begin{pmatrix} 8 \\ -3 \end{pmatrix}}$$

<u>Cramer's regel</u> $\quad x_1 = \frac{\Delta_1}{\Delta}, \; x_2 = \frac{\Delta_2}{\Delta}$

$$\Delta = \begin{vmatrix} 11 & -4 \\ -4 & 6 \end{vmatrix} = 66 - 16 = 50$$

$$\Delta_2 = \begin{vmatrix} 11 & 12 \\ -4 & -6 \end{vmatrix} = -66 + 48 = -18$$

$$\Delta_1 = \begin{vmatrix} 12 & -4 \\ -6 & 6 \end{vmatrix} = 72 - 24 = 48$$

$$x_1 = \frac{48}{50} = \frac{24}{25}$$

$$x_2 = \frac{-18}{50} = \frac{-9}{25}$$

ok!

4 $\begin{cases} x_1' = 2x_1 + 4x_2 \\ x_2' = -x_1 - 3x_2 \end{cases}$ eller $X' = A \cdot X$ där $A = \begin{bmatrix} 2 & 4 \\ -1 & -3 \end{bmatrix}$, $X = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

$|A - \lambda E| = 0$ eller $\begin{vmatrix} (2-\lambda) & 4 \\ -1 & (-3-\lambda) \end{vmatrix} = 0$ eller

$\lambda^2 + \lambda - 2 = 0 \Rightarrow \underline{\lambda_1 = -2,\ \lambda_2 = 1}$

$\underline{\lambda_1 = -2}$: $\begin{bmatrix} 4 & 4 \\ -1 & -1 \end{bmatrix} \sim \begin{array}{cc} v_1 & v_2 \\ \hline \end{array}\begin{bmatrix} 1 & 1 \end{bmatrix} \Rightarrow t \cdot \begin{pmatrix} -1 \\ 1 \end{pmatrix},\ t \neq 0$

$v_1 + v_2 = 0,\ v_2 = t,\ v_1 = -t$

$\underline{\lambda_2 = 1}$: $\begin{bmatrix} 1 & 4 \\ -1 & -4 \end{bmatrix} \sim \begin{array}{cc} v_1 & v_2 \\ \hline \end{array}\begin{bmatrix} 1 & 4 \end{bmatrix} \Rightarrow t \cdot \begin{pmatrix} -4 \\ 1 \end{pmatrix},\ t \neq 0$

$v_1 + 4v_2 = 0,\ v_2 = t,\ v_1 = -4t$

En bas av egenvektorer $\bar{p}_1 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$, $\bar{p}_2 = \begin{pmatrix} -4 \\ 1 \end{pmatrix}$

Svar den allmänna lösningen är

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = c_1 e^{-2 \cdot t} \cdot \bar{p}_1 + c_2 e^{1 \cdot t} \cdot \bar{p}_2 = c_1 e^{-2t} \begin{pmatrix} -1 \\ 1 \end{pmatrix} + c_2 e^{t} \begin{pmatrix} -4 \\ 1 \end{pmatrix}$$

$c_1, c_2 \in R$

(ii) $x_1(1) = -2,\ x_2(1) = 3$

$\begin{cases} -2 = -c_1 \cdot e^{-2} - 4c_2 \cdot e^{1} \\ 3 = c_1 \cdot e^{-2} + c_2 \cdot e^{1} \end{cases}$ | Inför: $c_1 e^{-2} = a$

$c_2 \cdot e^{1} = b$

⑥

$$\begin{cases} -2 = -a - 4b \\ 3 = a + b \end{cases} + \Rightarrow 1 = -3b \Rightarrow b = -1/3$$

$$\Rightarrow a = 3 - b = 3 + 1/3 = 10/3$$

dvs $c_1 e^{-2} = \frac{10}{3} \Rightarrow c_1 = \frac{10}{3} e^2$

$c_2 e^1 = -1/3 \Rightarrow c_2 = -\frac{1}{3} e^{-1}$

$$\Rightarrow \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \frac{10}{3} e^{-2(t-1)} \cdot \begin{pmatrix} -1 \\ 1 \end{pmatrix} - \frac{1}{3} e^{t-1} \cdot \begin{pmatrix} -4 \\ 1 \end{pmatrix}, \ t \in R$$

5 $4x_1^2 + 4x_1x_2 + 7x_2^2 = 3$

(i) $A = \begin{bmatrix} 4 & 2 \\ 2 & 7 \end{bmatrix}$, $|A - \lambda E| = 0$ dvs

$$\begin{vmatrix} (4-\lambda) & 2 \\ 2 & (7-\lambda) \end{vmatrix} = 0 \text{ dvs } \lambda^2 - 11\lambda + 24 = 0$$

$$\lambda_1 = 3, \ \lambda_2 = 8$$

$\lambda_1 = 3$: $\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \sim [1 \ 2] \Rightarrow t \cdot \begin{pmatrix} -2 \\ 1 \end{pmatrix}, \ t \neq 0$

$\lambda_2 = 8$: $\begin{bmatrix} -4 & 2 \\ 2 & -1 \end{bmatrix} \sim [1 \ -1/2] \Rightarrow t \cdot \begin{pmatrix} 1/2 \\ 1 \end{pmatrix} = s \cdot \begin{pmatrix} 1 \\ 2 \end{pmatrix}, s \neq 0$

$$\Rightarrow \bar{P}_1 = \frac{1}{\sqrt{5}} \begin{bmatrix} -2 \\ 1 \end{bmatrix}, \ \bar{P}_2 = \frac{1}{\sqrt{5}} \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \frac{1}{\sqrt{5}} \overbrace{\begin{pmatrix} -2 & 1 \\ 1 & 2 \end{pmatrix}}^{=P} \cdot \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \quad \text{eller}$$

$$\begin{cases} x_1 = \frac{1}{\sqrt{5}}(-2y_1 + y_2) \\ x_2 = \frac{1}{\sqrt{5}}(y_1 + 2y_2) \end{cases}$$

Obs P är en **ONmatris**

(ii) Insättning:

$$\frac{4}{5}(-2y_1 + y_2)^2 + \frac{4}{5}(-2y_1 + y_2)(y_1 + 2y_2) +$$

$$+ \frac{7}{5}(y_1 + 2y_2)^2 = 3 \quad \text{eller}$$

$$\frac{4}{5}(4y_1^2 - 4y_1y_2 + y_2^2) + \frac{4}{5}(-2y_1^2 \overbrace{- 4y_1y_2 + y_1y_2}^{-3y_1y_2}$$

$$+ 2y_2^2) + \frac{7}{5}(y_1^2 + 4y_1y_2 + 4y_2^2) = 3 \quad \text{eller}$$

$$y_1^2 \overbrace{\left(\frac{16}{5} - \frac{8}{5} + \frac{7}{5}\right)}^{=3} + y_1y_2 \overbrace{\left(\frac{-16}{5} - \frac{12}{5} + \frac{28}{5}\right)}^{=0}$$

$$+ y_2^2 \overbrace{\left(\frac{4}{5} + \frac{8}{5} + \frac{28}{5}\right)}^{=8} = 3 \quad \text{eller}$$

$$\boxed{3y_1^2 + 8y_2^2 = 3}$$

Kurvan är **en ellips**

(iii) $\min \lambda_1, \lambda_2 = 3 > 0 \Rightarrow Q$ är pos. def.

(6) Bevis eller motexempel

(i) $\det\left((A+B)^2\right) = (\det A + \det B)^2$

där A, B är s.a. $AB = BA$. (3×3) matriser

Ex: $A = B = E = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

$$V.L. = \det\begin{pmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{pmatrix} = 4^3 = 64$$

$$\#$$

$$H.L. = (\underset{1}{\underset{\shortparallel}{\det E}} + \underset{1}{\underset{\shortparallel}{\det E}})^2 = 2^2 = 4$$

(ii) $\bar{v}_1, \bar{v}_2$ är egenvektorer för A med egenvärde $\lambda_1 \neq \lambda_2$

Då är $\bar{v}_1, \bar{v}_2$ lin. oberoende.

Bevis: Sätt där $x_1\bar{v}_1 + x_2\bar{v}_2 = \bar{0}$ $(*)$

Använd $A(V.L.) = x_1\lambda_1\bar{v}_1 + x_2\lambda_2\bar{v}_2$

$A(H.L.) = A\bar{0} = 0$

$\Rightarrow x_1\lambda_1\bar{v}_1 + x_2\lambda_2\bar{v}_2 = \bar{0}$ (1)

Multiplicera med λ_1 $(*)$:

$x_1\lambda_1 v_1 + x_2\lambda_1\bar{v}_2 = \bar{0}$ (2)

(1) − (2): $x_2(\lambda_2 - \lambda_1)\bar{v}_2 = \bar{0}$

fg $\lambda_2 - \lambda_1 \neq 0$ o $\bar{v}_2 \neq \bar{0} \Rightarrow x_2 = 0 \Rightarrow (*)$

$x_1 \bar{v}_1 = \bar{0}$ og $\bar{v}_1 \neq \bar{0} \Rightarrow x_1 = 0$

d v s elv (*) har precis en lsg

$x_1 = 0$ o $x_2 = 0 \Rightarrow \bar{v}_1, \bar{v}_2$ är lin. oberoende.

(iii) $f(\bar{u}, \bar{v}) = u_1 v_2 + u_2 v_3 + u_3 \cdot v_1$

$\bar{u} = (u_1\ u_2\ u_3)$, $\bar{v} = (v_1\ v_2\ v_3)$

Obs 1) $f(\bar{u} + \bar{v}, \bar{w}) = f(\bar{u}, \bar{w}) + f(\bar{v}, \bar{w})$

2) $f(\lambda \bar{u} \cdot \bar{v}) = \lambda f(\bar{u}, \bar{v})$

3) $f(\bar{u}, \bar{v}) \neq f(\bar{v}, \bar{u})$

till ex. $\bar{u} = (1, 0, 0)$, $\bar{v} = (0, 1, 0)$

Ex: $f(\bar{u}, \bar{v}) = 1 \cdot 1 + 0 \cdot 0 + 0 \cdot 0 = 1$

$f(\bar{v}, \bar{u}) = 0 \cdot 0 + 1 \cdot 0 + 0 \cdot 1 = 0$

4) $f(\bar{u}, \bar{u}) = 0$ om $\bar{u} = (1, 0, 0)$ Obs