Rummet R^n, $n \geq 1$

$$\overline{v} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}, \quad -\overline{v} = \begin{pmatrix} -b_1 \\ -b_2 \\ \vdots \\ -b_n \end{pmatrix}, \quad \overline{0} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

Addition:

$$\overline{u} + \overline{v} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} + \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix} = \begin{pmatrix} x_1 + y_1 \\ x_2 + y_2 \\ \cdots \\ x_n + y_n \end{pmatrix}$$

Multiplikation med en konstant:

$$\lambda \cdot \overline{u} = \lambda \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} \lambda x_1 \\ \lambda x_2 \\ \lambda x_n \end{pmatrix}$$

Skalärprodukt:

$$\overline{u} \cdot \overline{v} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \cdot \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix} = x_1 y_1 + \ldots + x_n y_n$$

$$\Rightarrow |\overline{u}| = \sqrt{\overline{u} \cdot \overline{u}}$$

Obs Man kan införa en vinkel φ mellan $\overline{u}$ o $\overline{v}$:

$$\cos\varphi = \frac{\overline{u} \cdot v}{|\overline{u}| \cdot |\overline{v}|}$$

EX 1. $\bar{u} = (2, 3, 1, -1)$, $\bar{v} = (1, -2, 0, 5)$

Finn vinkeln mellan $\bar{u}$ o $\bar{v}$.

Lsg. $|\bar{u}| = \sqrt{\bar{u}\cdot\bar{u}} = \sqrt{\underset{\overset{\parallel}{4}}{2^2} + \underset{\overset{\parallel}{9}}{3^2} + \underset{\overset{\parallel}{1}}{1^2} + \underset{\overset{\parallel}{1}}{(-1)^2}} = \sqrt{15}$

$|\bar{v}| = \sqrt{\bar{v}\cdot\bar{v}} = \sqrt{\underset{\overset{\parallel}{1}}{1^2} + \underset{\overset{\parallel}{4}}{(-2)^2} + \underset{\overset{\parallel}{0}}{0^2} + \underset{\overset{\parallel}{25}}{5^2}} = \sqrt{30}$

$\bar{u}\cdot\bar{v} = \underbrace{2\cdot 1}_{2} + \underbrace{3\cdot(-2)}_{-6} + \underset{\overset{\parallel}{0}}{1\cdot 0} + \underset{\overset{\parallel}{-5}}{(-1)\cdot 5} = -9$

$\Rightarrow \cos\varphi = \dfrac{-9}{\sqrt{15}\cdot\sqrt{30}} = \dfrac{-9}{\sqrt{3\cdot 5}\,\sqrt{2\cdot 3\cdot 5}} = \dfrac{-9}{3\cdot 5\cdot\sqrt{2}} = -\dfrac{9}{15\sqrt{2}}$

Punkter i R^n:

$P\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$

Avståndet mellan A o B är $|\overline{AB}|$

EX 2 Finn avståndet mellan $A(2, 3, 4, 0, 1)$

$B(-1, 2, 3, 4, 1)$

Lsg. $\overline{AB} = B - A = (3, 1, 1, -4, 0)$

avståndet $= |\overline{AB}| = \sqrt{\underbrace{3^2}_{9} + \underbrace{1^2}_{1} + \underbrace{1^2}_{1} + \underbrace{(-4)^2}_{16} + \underbrace{0^2}_{0}} =$

$= \sqrt{27} = \underline{3\sqrt{3}}$

Allmänna vektorrum:

V, "$+$", "$\cdot$" är ett vektorrum om

f. a. gäller för alla $\bar{u}, \bar{v}, \bar{w}$ o $k, p \in R$.

(1) $\bar{u}+\bar{v}=\bar{v}+\bar{u}$ (2) $\bar{u}+(\bar{v}+\bar{w})=(\bar{u}+\bar{v})+\bar{w}$

(3) $\exists\ \bar{0}$ s.a. $\bar{u}+\bar{0}=\bar{0}+\bar{u}=\bar{u}$ (bara en!?)

(4) $\forall\ \bar{u}\ \exists\ -\bar{u}$ s.a. $\bar{u}+(-\bar{u})=(-\bar{u})+\bar{u}=\bar{0}$
(bara en!?)

(5) $k\cdot(\bar{u}+\bar{v})=k\cdot\bar{u}+k\cdot\bar{v}$ 6) $(k+p)\cdot\bar{u}=k\cdot\bar{u}+p\cdot\bar{u}$

(7) $k\cdot(p\cdot\bar{u})=(kp)\cdot\bar{u}$ 8) $1\cdot\bar{u}=\bar{u}$

Ex 3. Låt $V=\{f:\ f$ är kontinuerlig på $[0,1]\}$

"$+$" $(f+g)(x)=f(x)+g(x)\ \forall x$; "$\cdot$" $(\lambda\cdot f)(x)=\lambda\cdot f(x)\ \forall x$

Visa att V är ett vektorrum.

En skalärprodukt på ett vektorrum V

$\bar{u}\cdot\bar{v}\in R$:

(1) $\bar{u}\cdot\bar{v}=\bar{v}\cdot\bar{u}$ (2) $\bar{u}\cdot(\bar{v}+\bar{w})=\bar{u}\cdot\bar{v}+\bar{u}\cdot\bar{w}$

(3) $\lambda(\bar{u}\cdot\bar{v})=(\lambda\bar{u})\cdot\bar{v}$ (4) $\bar{u}\cdot\bar{u}\geq 0$ o

$\bar{u}\cdot\bar{u}=0 \Leftrightarrow \bar{u}=\bar{0}$.

Ex 4. Vektorrum från Ex 3.

Inför $(f \cdot g) = 5 \cdot \int_0^1 f(x) \cdot g(x)\,dx$

Finn $|f|$ där $f(x) = x^2 + x$, $x \in [0,1]$

Lsg. $|f| = \sqrt{f \cdot f}$.

$$f \cdot f = 5\int_0^1 (x^2+x)^2\,dx = 5\int_0^1 (x^4 + 2x^3 + x^2)\,dx =$$

$$= 5\left(\frac{x^5}{5} + 2 \cdot \frac{x^4}{4} + \frac{x^3}{3}\right)\Big|_0^1 = 5 \cdot \left(\frac{1}{5} + \frac{1}{2} + \frac{1}{3}\right)$$

$$= 1 + 2.5 + 5/3 = \frac{7}{2} + \frac{5}{3} = \frac{31}{6} \Rightarrow |f| = \sqrt{\frac{31}{6}}.$$