Instructions: Please attempt all questions. You may answer either in English or Swedish. There are five questions, each worth 16 points. To obtain a grade 3, 4 or 5, you must obtain at least 40, 48 or 56 points (50%, 60% or 70%) respectively. You may not use any notes, textbooks or electronic devices. Good luck!

Svara på alla uppgifter. Du får svara antingen på engelska eller svenska. Det finns fem uppgifter och varje uppgift kan ge maximalt 16 poäng. För att få betyg 3, 4 eller 5 kravs minst 40, 48 respektive 56 poäng (50%, 60% respektive 70%). Inga hälpmedel tillåtna. Lycka till!

(1) Let $f\colon \mathbf{R}^n \times (0,\infty) \to \mathbf{R}$ and $g\colon \mathbf{R}^n \to \mathbf{R}$ be two smooth functions and $b \in \mathbf{R}^n$. Consider the equations

$$\begin{aligned} u_t(x,t) + b\cdot\nabla u(x,t) &= f(x,t) \quad \text{for } x\in\mathbf{R}^n \text{ and } t>0, \text{ and} \\ u(x,0) &= g(x) \quad \text{for } x\in\mathbf{R}^n. \end{aligned} \tag{$\clubsuit$}$$

Here ∇ denotes the gradient vector in the x-variables.

(a) Show that the characteristic curves (X,T) of the homogeneous equation $v_t(x,t) + b\cdot\nabla v(x,t) = 0$ are given by $X(s) = x + bs$ and $T(s) = t + s$. **[4 marks]**

(b) Let u be a solution to ($\clubsuit$). Set $z(s) = u(x+bs, t+s)$ for fixed $x\in\mathbf{R}^n$ and $t>0$ and derive an ordinary differential equation which z satisfies. **[6 marks]**

(c) Use the ordinary differential equation from (b) to derive the formula

$$u(x,t) = g(x-bt) + \int_0^t f(x-bs, t-s)ds.$$

for a solution u to ($\clubsuit$). **[6 marks]**

(2) Suppose Ω is a connected bounded open subset of $\mathbf{R}^n$. Recall that the Weak Maximum Principle for harmonic functions reads as follows: If $u\colon \overline{\Omega} \to \mathbf{R}$ is a continuous function which is harmonic in Ω, then the maximum and minimum values of u are attained on $\partial\Omega$.

(a) State what it means for u to be harmonic in Ω. **[3 marks]**

(b) Prove the Weak Maximum Principle for harmonic functions. You may find it helpful to use the auxillary function $v(\mathbf{x}) = u(\mathbf{x}) + \varepsilon|\mathbf{x}|^2$. **[7 marks]**

(c) Use the Weak Maximum Principle to prove that there is at most one continuous function $v\colon \overline{\Omega} \to \mathbf{R}$ such that

$$\begin{cases} \Delta v = f & \text{in } \Omega, \text{ and} \\ v = g & \text{on } \partial\Omega, \end{cases}$$

for given functions $f\colon \Omega \to \mathbf{R}$ and $g\colon \partial\Omega \to \mathbf{R}$. **[6 marks]**

(3) Consider the following initial value problem for the wave equation:

$$\begin{cases} \partial_t^2 u(x,t) - \partial_x^2 u(x,t) = 0 & \text{for } x \in \mathbf{R} \text{ and } t > 0, \\ u(x,0) = g(x) \quad \text{and} \quad \partial_t u(x,0) = h(x) & \text{for } x \in \mathbf{R}. \end{cases} \tag{$\diamondsuit$}$$

Recall D'Alembert's formula is

$$u(x,t) = \frac{1}{2}\left(g(x+t) + g(x-t)\right) + \frac{1}{2}\int_{x-t}^{x+t} h(y)dy. \tag{$\heartsuit$}$$

(a) Prove that if $g \in C^2(\mathbf{R})$ and $h \in C^1(\mathbf{R})$, then ($\heartsuit$) solves ($\diamondsuit$). **[6 marks]**

(b) Recall that the energy of a solution u to ($\diamondsuit$) is

$$E[u](t) = \frac{1}{2}\int_{-\infty}^{\infty} (\partial_t u(x,t))^2 + (\partial_x u(x,t))^2 dx.$$

(i) Prove that $E[u]$ is a constant function. (You may assume that all integrals you calculate with converge uniformly in t and $u(x,t) \to 0$ as $x \to \pm\infty$.) **[6 marks]**

(ii) Use part (i) to prove there is at most one solution to ($\diamondsuit$). **[4 marks]**

(4) The aim of this question is to solve the initial value problem

$$\begin{cases} \partial_t u(x,t) - \partial_{xx} u(x,t) = 0 & \text{for } x \in \mathbf{R} \text{ and } t \in (0,\infty); \\ u(x,0) = \phi(x) & \text{for } x \in \mathbf{R}, \end{cases} \tag{$\spadesuit$}$$

with initial data

$$\phi(x) = \begin{cases} 1 & \text{if } x > 0; \\ \frac{1}{2} & \text{if } x = 0; \\ 0 & \text{if } x < 0. \end{cases}$$

(a) We begin by looking for a solution of the form

$$u(x,t) = g(x/(2\sqrt{t}))$$

for some $g\colon \mathbf{R} \to \mathbf{R}$. Show that if any such u solves ($\spadesuit$) then g solves the ordinary differential equation

$$g''(p) + 2pg'(p) = 0.$$

[6 marks]

(b) Find the general formula for solutions g of the ordinary differential equation above. Use this formula for g together with the initial data ϕ to find the solution u of ($\spadesuit$). You may use the fact $\int_0^\infty e^{-q^2} dq = \int_{-\infty}^0 e^{-q^2} dq = \sqrt{\pi}/2$ without proof. **[7 marks]**

(c) Observe that $\partial_x u(x,t)$ is equal to $S(x,t)$, the heat kernel. Can you give some motivation for why this is so? **[3 marks]**

(5) Fix $\delta x > 0$. For a function $u\colon \mathbf{R} \to \mathbf{R}$ which is four times continuously differentiable, define $x_j = j\delta x$ and set $u_j = u(x_j)$.

(a) Define the forward, backward and centred difference of u at x_j. **[6 marks]**

(b) Use Taylor's theorem to estimate the error between the differences you defined above and the first derivative $u'(x_j)$. **[10 marks]**