**Instructions:** Please attempt all questions. You may answer either in English or Swedish. There are five questions, each worth 16 points. To obtain a grade 3, 4 or 5, you must obtain at least 40, 48 or 56 points (50%, 60% or 70%) respectively. You may not use any notes, textbooks or electronic devices. Good luck!

Svara på alla uppgifter. Du får svara antingen på engelska eller svenska. Det finns fem uppgifter och varje uppgift kan ge maximalt 16 poäng. För att få betyg 3, 4 eller 5 krävs minst 40, 48 respektive 56 poäng (50%, 60% respektive 70%). Inga hälpmedel tillåtna. Lycka till!

(1) Use the method of characteristics to find a smooth function $u\colon \mathbf{R}^2 \to \mathbf{R}$ which solves the equation

$$xu_x(x,t) + u_t(x,t) + 2u(x,t) = 0 \quad \text{for all } (x,t) \in \mathbf{R}^2$$

and satisfies the condition $u(x,0) = \ln(1+x^2)$ for all $x \in \mathbf{R}$. **[16 marks]**

(2) Let $\Omega \subset \mathbf{R}^n$ be a bounded open set and $\mathbf{b} \in \mathbf{R}^n$ be a vector which satisfies $\mathbf{b}\cdot\mathbf{x} + n > 0$ for all $\mathbf{x} \in \Omega$.

(a) Prove that continuous functions $u\colon \overline{\Omega} \to \mathbf{R}$ which solve

$$\Delta u(\mathbf{x}) + \mathbf{b} \cdot \nabla u(\mathbf{x}) = 0$$

for $\mathbf{x} \in \Omega$ satisfy the weak maximum principle:

$$\max_{\overline{\Omega}} u = \max_{\partial\Omega} u.$$

[Hint: The function $\mathbf{x} \mapsto \varepsilon|\mathbf{x}|^2$ for $\varepsilon > 0$ may be useful.] **[10 marks]**

(b) Suppose a continuous function $g\colon \partial\Omega \to \mathbf{R}$ is given. Prove that there cannot exist more than one continuous function $u\colon \overline{\Omega} \to \mathbf{R}$ which solves the boundary value problem

$$\begin{cases} \Delta u + \mathbf{b} \cdot \nabla u = 0 & \text{in } \Omega; \\ u = g & \text{on } \partial\Omega. \end{cases}$$

**[6 marks]**

(3) Let $\Omega$ be an open set and $\phi\colon \Omega \to \mathbf{R}$. Consider the initial boundary value problem

$$\begin{cases} \partial_t u(\mathbf{x},t) - \Delta u(\mathbf{x},t) = 0 & \text{for } \mathbf{x} \in \Omega \text{ and } t \in (0,T]; \\ u(\mathbf{x},0) = \phi(x) & \text{for } \mathbf{x} \in \Omega; \text{ and} \\ u(\mathbf{y},t) = 0 & \text{for } \mathbf{y} \in \partial\Omega \text{ and } t \in (0,T]. \end{cases} \tag{$\spadesuit$}$$

(a) Show that

$$\int_\Omega |u(\mathbf{x},t)|^2 d\mathbf{x}$$

is a decreasing function of $t$ for each $u \in C^2(\overline{\Omega} \times [0,T])$ which solves ($\spadesuit$). **[8 marks]**

(b) Use (a) to prove there cannot exist more than one function $u \in C^2(\overline{\Omega} \times [0,T])$ which solves ($\spadesuit$). **[8 marks]**

(4) For a smooth solution $u$ of the wave equation $\partial_{tt}u(x,t) - \partial_{xx}u(x,t) = 0$, the energy density is defined to be

$$e(x,t) = \frac{1}{2}((\partial_t u(x,t))^2 + (\partial_x u(x,t))^2)$$

and the momentum density

$$p(x,t) = \partial_t u(x,t) \partial_x u(x,t).$$

(a) Show that $\partial e/\partial t = \partial p/\partial x$ and $\partial p/\partial t = \partial e/\partial x$. **[8 marks]**

(b) Show that $e$ and $p$ also satsify the wave equation. **[8 marks]**

(5) Suppose that a solution $u$ to the Schrödinger equation

$$-i\partial_t u(x,t) = \partial_{xx}u(x,t) - x^2 u(x,t)$$

is of the form $u(x,t) = T(t)v(x)$.

(a) Show that $v$ satisfies the equation

$$v''(x) + (\lambda - x^2)v(x) = 0, \tag{$\heartsuit$}$$

for some constant $\lambda$. **[6 marks]**

(b) We saw in lectures that, by performing the substitution $v(x) = w(x)e^{x^2/2}$, it is possible to show ($\heartsuit$) is equivalent to

$$w''(x) - 2xw'(x) + (\lambda - 1)w(x) = 0. \tag{$\diamondsuit$}$$

Show that if $w$ is a power series, that is $w(x) = \sum_{k=0}^{\infty} a_k x^k$, then we must have

$$(k+2)(k+1)a_{k+2} = (2k+1-\lambda)a_k \quad \text{for each } k.$$

**[6 marks]**

(c) Find a polynomial solution $w$ to ($\diamondsuit$) when $\lambda = 9$. **[4 marks]**