Instructions: Please attempt all questions. You may answer either in English or Swedish. There are five questions, each worth 16 points. To obtain a grade 3, 4 or 5, you must obtain at least 40, 48 or 56 points (50%, 60% or 70%) respectively. You may not use any notes, textbooks or electronic devices. Good luck!

Svara på alla uppgifter. Du får svara antingen på engelska eller svenska. Det finns fem uppgifter och varje uppgift kan ge maximalt 16 poäng. För att få betyg 3, 4 eller 5 kravs minst 40, 48 respektive 56 poäng (50%, 60% respektive 70%). Inga hälpmedel tillåtna. Lycka till!

(1) Consider the partial differential equation

$$\partial_x u(x,y) + 2xy^2 \partial_y u(x,y) = 0 \quad \text{for } x, y \in \mathbf{R}. \tag{$\spadesuit$}$$

(a) What is the order of the equation? Is it linear or non-linear? **[4 marks]**

(b) Show that any solution to ($\spadesuit$) is constant on the curves in the xy-plane given by

$$y = \frac{1}{C - x^2}.$$

for any constant C. **[6 marks]**

(c) Is it possible to find a solution to ($\spadesuit$) such that $u(0,y) = \sin y$ for all $y \in \mathbf{R}$? Is there a solution such that $u(x,1) = \sin x$? **[6 marks]**

(2) Recall the Poisson formula

$$u(\mathbf{x}) = \frac{(a^2 - |\mathbf{x}|^2)}{2\pi a} \int_{|\mathbf{y}|=a} \frac{\tilde{h}(\mathbf{y})}{|\mathbf{x}-\mathbf{y}|^2} d\sigma(\mathbf{y}) \tag{$\heartsuit$}$$

gives the solution to the boundary value problem

$$\begin{cases} \Delta u = 0 & \text{in } B_a(\mathbf{0}), \text{ and} \\ u = h & \text{on } \partial B_a(\mathbf{0}), \end{cases}$$

where $B_a(\mathbf{0}) = \{\mathbf{x} \in \mathbf{R}^2 \mid |\mathbf{x} - \mathbf{0}| < a\}$ is the ball in the plane of radius a centred at the origin $\mathbf{0}$.

(a) For a harmonic function $u\colon \Omega \to \mathbf{R}$, where $\Omega \subset \mathbf{R}^2$ is an open set, state the mean value property u satisfies. **[4 marks]**

(b) Use the Poisson formula above to prove the mean value property you stated in part (a). **[6 marks]**

(c) Suppose $u\colon \overline{B_a(\mathbf{0})} \to \mathbf{R}$ is harmonic in $B_a(\mathbf{0})$ and such that $u(a,\theta) = \sin^2\theta + 2$ on $\partial B_a(\mathbf{0})$ (writen in polar coordinates). Without calculating the solution explicity, compute the maximum value of u in $\overline{B_a(\mathbf{0})}$ and the value of u at the origin. **[6 marks]**

(3) Consider the following initial value problem for the wave equation:

$$\begin{cases} \partial_t^2 u(x,t) - \partial_x^2 u(x,t) = 0 & \text{for } x \in \mathbf{R} \text{ and } t > 0, \\ u(x,0) = g(x) \quad \text{and} \quad \partial_t u(x,0) = h(x) & \text{for } x \in \mathbf{R}. \end{cases} \qquad (\diamondsuit)$$

Recall D'Alembert's formula is

$$u(x,t) = \frac{1}{2}\left(g(x+t) + g(x-t)\right) + \frac{1}{2}\int_{x-t}^{x+t} h(y)dy.$$

(a) Calculate u when $g(x) = \ln(1+x^2)$ and $h(x) = 4 + x$. **[6 marks]**

(b) Recall that the energy of a solution u to $(\diamondsuit)$ is

$$E[u](t) = \frac{1}{2}\int_{-\infty}^{\infty} (\partial_t u(x,t))^2 + (\partial_x u(x,t))^2 dx.$$

(i) Prove that $E[u]$ is a constant function. (You may assume that all integrals you calculate with converge uniformly in t and $u(x,t) \to 0$ as $x \to \pm\infty$.) **[6 marks]**

(ii) Use part (i) to prove there is at most one solution to $(\diamondsuit)$. **[4 marks]**

(4) Suppose Ω is a connected bounded open set. Recall that the Weak Maximum Principle for the heat equation reads as follows: Let $u\colon \overline{\Omega}\times[0,T] \to \mathbf{R}$ be a continuous function which is also a solution to the heat equation $\partial_t u(\mathbf{x},t) - \Delta u(\mathbf{x},t) = 0$ for $(\mathbf{x},t) \in \Omega \times (0,T]$. Then the maximum value of u is attained at a point $(\mathbf{x},t) \in \overline{\Omega} \times [0,T]$ such that either $t = 0$ or $\mathbf{x} \in \partial\Omega$.

(a) Sketch the the domain of u and indicate the set where the Weak Maximum Principle shows the maximum of u will be attained. **[3 marks]**

(b) Prove the Weak Maximum Principle stated above. You may find it helpful to use the auxillary function $v(\mathbf{x},t) = u(\mathbf{x},t) + \varepsilon|\mathbf{x}|^2$. **[6 marks]**

(c) Using the Weak Maximum Principle, prove that the minimum of u will be attained on the same set. **[4 marks]**

(d) Using the above prove that there is at most one continuous function $v\colon \overline{\Omega} \to \mathbf{R}$ such that

$$\begin{cases} \partial_t v(\mathbf{x},t) - \Delta v(\mathbf{x},t) = f(\mathbf{x},t) & \text{for } \mathbf{x} \in \Omega \text{ and } t \in (0,T]; \\ v(\mathbf{x},0) = \phi(x) & \text{for } \mathbf{x} \in \overline{\Omega}; \text{ and} \\ v(\mathbf{y},t) = g(\mathbf{y},t) & \text{for } \mathbf{y} \in \partial\Omega \text{ and } t \in (0,T]. \end{cases}$$

for given functions $f\colon \Omega \times (0,T] \to \mathbf{R}$, $\phi\colon \overline{\Omega} \to \mathbf{R}$ and $g \times (0,T]\colon \partial\Omega \to \mathbf{R}$. **[3 marks]**

(5) Fix $\delta x > 0$ and $\delta t > 0$ and set $s = (\delta t)/(\delta x)^2$. For a function $u\colon \mathbf{R} \times [0, \infty) \to \mathbf{R}$ define $x_j = j\delta x$ and $t_n = n\delta t$, and set $u_j^n = u(x_j, t_n)$.

Fix $J \in \mathbf{N}$. We consider the following numerical scheme for approximating the heat equation with Dirichlet boundary data on $[0, x_J] \times [0, \infty)$: $u_j^0 = \phi_j$ are given for $j = 1, 2, \ldots, J$ and $u_0^n = u_J^n = 0$ for $n > 0$. The remaining values are determined by

$$u_j^{n+1} = (u_{j+1}^n + u_{j-1}^n)s + (1 - 2s)u_j^n.$$

(a) Suppose $u_j^n = X_j T^n$ and derive the two difference equations

$$T_{n+1} = \xi T_n$$

and

$$s\frac{X_{j+1} + X_{j-1}}{X_j} + 1 - 2s = \xi,$$

for a constant ξ. **[5 marks]**

(b) Find explicit expressions for the solutions T_n and X_j to the above equations which also satisfy the conditions $X_0 = X_J = 0$ and show that $\xi = 1 - 2s(1 - \cos(k\pi/J)$ for some $k \in \mathbf{Z}$. **[7 marks]**

(c) Use your calculation above to justify the stability condition $(\delta t)/(\delta x)^2 = s \leq 1/2$. **[4 marks]**