TATA27 **14th August 2017**

Partial Differential Equations

Comment: *All definitions and theorems have been given in lectures. The remaining content is similar to given homework questions or is work done in lectures.*

1. Use the method of characteristics to find a smooth function $u\colon \mathbf{R}^2 \to \mathbf{R}$ which solves the equation

$$xu_x(x,t) + u_t(x,t) + 8u(x,t) = 0 \quad \text{for all } (x,t) \in \mathbf{R}^2$$

and satisfies the condition $u(x,0) = (x+2)/(1+x^2)$ for all $x \in \mathbf{R}$. **[16 marks]**

Solution:

We search for appropriate curves (X,T) such that the solution on the curves $s \mapsto z(s) := u(X(s),T(s))$ behaves nicely. We have

$$z'(s) = \frac{d}{ds}u(X(s),T(s)) = X'(s)\partial_1 u(X(s),T(s)) + T'(s)\partial_2 u(X(s),T(s)),$$

so it seems reasonable to set $X'(s) = X(s)$ and $T'(s) = 1$. Thus $X(s) = c_X e^s$ and $T(s) = s + c_T$ for constants $c_X, c_T \in \mathbf{R}$. We can then rewrite the PDE as

$$z'(s) + 8z(s) = X(s)\partial_1 u(X(s),T(s)) + \partial_2 u(X(s),T(s)) + 8u(X(s),T(s)) = 0.$$

This is an ODE with general solution $z(s) = Ae^{-8s}$ for arbitrary $A \in \mathbf{R}$.

Now fix (x,t). If we choose $c_X = x$ and $c_T = t$, then $X(s) = xe^s$ and $T(s) = s + t$, and when $s = 0$ the characteristic curve passes through $(X(0),T(0)) = (x,t)$ and when $s = -t$ the curve passes through $(X(-t),T(-t)) = (xe^{-t},0)$. When $s = -t$ we can use the initial condition to find the value of z:

$$z(-t) = u(X(-t),T(-t)) = u(xe^{-t},0) = \frac{xe^{-t}+2}{1+(xe^{-t})^2}.$$

But on the other hand, using the form of the general solution to the characteristic ODE, $z(-t) = Ae^{8t}$, so $A = e^{-8t}(xe^{-t}+2)/(1+(xe^{-t})^2))$. Equally, for $s = 0$,

$$u(x,t) = z(0) = Ae^{-2\times 0} = A = e^{-8t}\frac{xe^{-t}+2}{1+(xe^{-t})^2} = \frac{xe^{-9t}+2e^{-8t}}{1+x^2e^{-2t}},$$

which gives us an expression for the solution u at (x,t).

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2. Let $\Omega \subset \mathbf{R}^n$ be a bounded open set and $\mathbf{b} \in \mathbf{R}^n$ be a vector which satisfies $\mathbf{b}\cdot\mathbf{x} + n > 0$ for all $\mathbf{x} \in \Omega$.

(a) Prove that continuous functions $u\colon \overline{\Omega} \to \mathbf{R}$ which solve

$$\Delta u(\mathbf{x}) + \mathbf{b}\cdot\nabla u(\mathbf{x}) = 0$$

for $\mathbf{x} \in \Omega$ satisfy the weak maximum principle:

$$\max_{\overline{\Omega}} u = \max_{\partial\Omega} u.$$

[Hint: The function $\mathbf{x} \mapsto \varepsilon|\mathbf{x}|^2$ for $\varepsilon > 0$ may be useful.] **[10 marks]**

(b) Suppose a continuous function $g\colon \partial\Omega \to \mathbf{R}$ is given. Prove that there cannot exist more than one continuous function $u\colon \overline{\Omega} \to \mathbf{R}$ which solves the boundary value problem

$$\begin{cases} \Delta u + \mathbf{b} \cdot \nabla u = 0 & \text{in } \Omega; \\ u = g & \text{on } \partial\Omega. \end{cases}$$

[6 marks]

Solution:

(a) For $\varepsilon > 0$ set $v(\mathbf{x}) = u(\mathbf{x}) + \varepsilon|\mathbf{x}|^2$. As the sum of two continuous functions, v is continuous on $\overline{\Omega}$ and so must attain a maximum somewhere in the compact set $\overline{\Omega} = \Omega \cup \partial\Omega$. We will now rule out the possibility that v attains its maximum in Ω. Suppose to the contrary that v attains this maximum $\mathbf{x} \in \Omega$. Then we know $\mathbf{x}$ is a critical point, so $\nabla v(\mathbf{x}) = 0$ and, by the second derivative test, $\Delta v(\mathbf{x}) = \sum_{j=1}^{n} \partial_j^2 v(\mathbf{x}) \leq 0$. Therefore

$$\Delta v(\mathbf{x}) + \mathbf{b} \cdot \nabla v(\mathbf{x}) = \Delta v(\mathbf{x}) + 0 \leq 0 + 0 = 0.$$

But on the other hand, we can compute

$$\Delta v(\mathbf{x}) + \mathbf{b} \cdot \nabla v(\mathbf{x}) = \Delta u(\mathbf{x}) + \mathbf{b} \cdot \nabla u(\mathbf{x}) + 2\varepsilon n + 2\varepsilon \mathbf{b} \cdot \mathbf{x} \geq 2\varepsilon(\mathbf{b} \cdot \mathbf{x} + n) > 0,$$

via the differential equality u satisfies and the condition on $\mathbf{b}$. These two inequalities contradict each other, so v cannot attain its maximum in Ω.

Therefore v must attain its maximum at a point $\mathbf{y} \in \partial\Omega$. Thus, for any $\mathbf{x} \in \overline{\Omega}$,

$$u(\mathbf{x}) \leq v(\mathbf{x}) \leq v(\mathbf{y}) = u(\mathbf{y}) + \varepsilon|\mathbf{y}|^2 \leq u(\mathbf{y}) + \varepsilon C^2 \leq \max_{\partial\Omega} u + \varepsilon C^2,$$

where C is the constant obtained from the fact Ω is bounded. Since the above inequality holds for any $\varepsilon > 0$, we have $u(\mathbf{x}) \leq \max_{\partial\Omega} u$ for any $\mathbf{x} \in \overline{\Omega}$, so

$$\max_{\overline{\Omega}} u \leq \max_{\partial\Omega} u$$

Because $\partial\Omega \subseteq \overline{\Omega}$ we have that $\max_{\partial\Omega} u \leq \max_{\overline{\Omega}} u$ and combining these two inequalities we get that $\max_{\overline{\Omega}} u = \max_{\partial\Omega} u$ and the maximum of u is attained on $\partial\Omega$.

(b) Suppose we had two solutions u_1 and u_2. Then $u := u_2 - u_1$ satisfies the same equation, but has the boundary value 0. Thus

$$\max_{\overline{\Omega}} u = \max_{\partial\Omega} u = \max_{\partial\Omega} 0 = 0.$$

Thus $u_2 - u_1 = u \leq 0$ and so $u_2 \leq u_1$. Swapping the roles of u_1 and u_2 we find that $u_1 \leq u_2$. Hence $u_1 = u_2$ and there can be at most one continuous solution.

3.

Consider a solution u to the *damped string equation*

$$\partial_{tt}u(x,t) - c^2\partial_{xx}u(x,t) + r\partial_t u(x,t) = 0 \quad (x \in \mathbf{R},\, t > 0)$$

for $c^2 = T/\rho$ and given constants $T, \rho, r > 0$. Define the energy of a solution u at time t by the formula

$$E[u](t) = \frac{1}{2}\int_{-\infty}^{\infty} \rho(\partial_t u(x,t))^2 + T(\partial_x u(x,t))^2 dx.$$

(a) Assuming u and its derivatives are sufficiently smooth and tend to zero as $x \to \pm\infty$, show that the energy $E[u]$ is a non-increasing function.

[8 marks]

(b) Prove that there cannot exist more than one solution u to the damped string equation which satisfies the same assumptions you made in (a) together with the initial conditions $u(x,0) = f(x)$ and $\partial_t u(x,0) = g(x)$, for given smooth functions f and g. **[8 marks]**

Solution:

(a) We have

$$\begin{aligned}
\frac{d}{dt}\left(\frac{1}{2}\int_{-\infty}^{\infty} \rho(\partial_t u(x,t))^2 dx\right) &= \frac{1}{2}\int_{-\infty}^{\infty} \rho\partial_t u(x,t)\partial_{tt}u(x,t)dx \\
&= \frac{1}{2}\int_{-\infty}^{\infty} \partial_t u(x,t)(T\partial_{xx}u(x,t) - r\rho\partial_t u(x,t))dx \\
&= -\frac{1}{2}\int_{-\infty}^{\infty} \partial_{tx}u(x,t)\partial_x u(x,t)dx - \frac{1}{2}\int_{-\infty}^{\infty} r\rho(\partial_t u(x,t))^2 dx \\
&= -\frac{d}{dt}\left(\frac{1}{2}\int_{-\infty}^{\infty} T(\partial_x u(x,t))^2 dx\right) - \frac{1}{2}\int_{-\infty}^{\infty} r\rho(\partial_t u(x,t))^2 dx.
\end{aligned}$$

Therefore

$$E[u]'(t) = \frac{d}{dt}\left(\frac{1}{2}\int_{-\infty}^{\infty} \rho(\partial_t u(x,t))^2 + T(\partial_x u(x,t))^2 dx\right) = -\frac{1}{2}\int_{-\infty}^{\infty} r\rho(\partial_t u(x,t))^2 dx \leq 0,$$

hence the energy $E[u]$ is a non-increasing function.

(b) Suppose there exists two such solutions, u_1 and u_2. Then $w = u_2 - u_2$ also solves the damped string equation with initial conditions $w(x,0) = u_2(x,0) - u_1(x,0) = f(x) - f(x) = 0$ and $\partial_t w(x,0) = \partial_t u(x,0) - \partial_t u(x,0) = g(x) - g(x) = 0$. Thus

$$E[w](0) = \frac{1}{2}\int_{-\infty}^{\infty} \rho(0)^2 + T(0)^2 dx = 0.$$

Moreover, $E[w](t)$ is a non-negative integral which is non-increasing in time t, so we must have $E[w](t) = 0$ for all $t \geq 0$. This together with the smoothness of w implies, in particular, that $\partial_x w(x,t) = 0$ for all x and t, and so we know that $x \mapsto w(x,t)$ must be a constant function for each fixed t. But since $w(x,t) \to 0$ as $x \to \pm\infty$, we conclude $w(x,t) = 0$ for all x and t. Thus $u_1 = u_2$ and there cannot exist two distinct solutions.

4. Let Ω be an open set with C^1 boundary and $h\colon \partial\Omega \to \mathbf{R}$ a C^1 function. Define the energy of each continuously differentiable $v\colon \Omega \to \mathbf{R}$ to be

$$E_h[v] = \frac{1}{2}\int_\Omega |\nabla v(\mathbf{x})|^2 d\mathbf{x} - \int_{\partial\Omega} h(\mathbf{x})v(\mathbf{x})d\sigma(\mathbf{x}).$$

Show that a function $u \in C^2(\overline{\Omega})$ which satisfies the boundary value problem

$$\begin{cases} \Delta u = 0 & \text{in } \Omega, \text{ and} \\ \frac{\partial u}{\partial \mathbf{n}} := \mathbf{n}\cdot\nabla u = h & \text{on } \partial\Omega \end{cases}$$

is such that

$$E_h[u] \le E_h[v]$$

for all $v \in C^1(\overline{\Omega})$. Here $\mathbf{n}$ is the outward unit normal to $\partial\Omega$. **[16 marks]**

Solution: Suppose $u \in C^2(\overline{\Omega})$ satisfies the boundary value problem

$$\begin{cases} \Delta u = 0 & \text{in } \Omega, \text{ and} \\ \frac{\partial u}{\partial \mathbf{n}} := \mathbf{n}\cdot\nabla u = h & \text{on } \partial\Omega \end{cases}$$

For $v \in C^1(\overline{\Omega})$ set $w = v - u$. Then, using Green's first identity,

$$\begin{aligned} E_h[v] &= \frac{1}{2}\int_\Omega |\nabla v(\mathbf{x})|^2 d\mathbf{x} - \int_{\partial\Omega} h(\mathbf{x})v(\mathbf{x})d\sigma(\mathbf{x}) \\ &= \frac{1}{2}\int_\Omega |\nabla w(\mathbf{x}) + \nabla u(\mathbf{x})|^2 d\mathbf{x} - \int_{\partial\Omega} h(\mathbf{x})(w(\mathbf{x}) + u(\mathbf{x}))d\sigma(\mathbf{x}) \\ &= \frac{1}{2}\int_\Omega |\nabla w(\mathbf{x})|^2 d\mathbf{x} + E_h[u] + \int_\Omega \nabla w(\mathbf{x})\cdot\nabla u(\mathbf{x})d\mathbf{x} - \int_{\partial\Omega} h(\mathbf{x})w(\mathbf{x})d\sigma(\mathbf{x}) \\ &= \frac{1}{2}\int_\Omega |\nabla w(\mathbf{x})|^2 d\mathbf{x} + E_h[u] - \int_\Omega w(\mathbf{x})\Delta u(\mathbf{x})d\mathbf{x} + \int_{\partial\Omega} \left(\frac{\partial u}{\partial \mathbf{n}}(\mathbf{x}) - h(\mathbf{x})\right) w(\mathbf{x})d\sigma(\mathbf{x}) \\ &= \frac{1}{2}\int_\Omega |\nabla w(\mathbf{x})|^2 d\mathbf{x} + E_h[u] \\ &\ge E_h[u]. \end{aligned}$$

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5. Suppose that a solution u to the Schrödinger equation

$$-i\partial_t u(x,t) = \partial_{xx}u(x,t) - x^2 u(x,t)$$

is of the form $u(x,t) = T(t)v(x)$.

(a) Show that v satisfies the equation

$$v''(x) + (\lambda - x^2)v(x) = 0, \qquad (\heartsuit)$$

for some constant λ. **[6 marks]**

(b) We saw in lectures that, by performing the substitution $v(x) = w(x)e^{x^2/2}$, it is possible to show ($\heartsuit$) is equivalent to

$$w''(x) - 2xw'(x) + (\lambda - 1)w(x) = 0. \qquad (\diamondsuit)$$

Show that if w is a power series, that is $w(x) = \sum_{k=0}^{\infty} a_k x^k$, then we must have

$$(k+2)(k+1)a_{k+2} = (2k+1-\lambda)a_k \quad \text{for each } k.$$

[6 marks]

(c) Find a polynomial solution w to ($\diamondsuit$) when $\lambda = 9$. **[4 marks]**

Solution:

(a) Substituting $u(x,t) = T(t)v(x)$ in Schrödinger's equation and dividing by $T(t)v(x)$ gives

$$-i\frac{T'(t)}{T(t)} = \frac{v''(x) - x^2}{v(x)}$$

Since the left-hand side only depends on t and the right-hand side only depends on x, both must be equal to $-\lambda$, say, for some constant λ. Therefore

$$v''(x) + (\lambda - x^2)v(x) = 0,$$

for some constant λ.

(b) Substituting this power series in ($\diamondsuit$) we get

$$\sum_{k=0}^{\infty} k(k-1)a_k x^{k-2} - \sum_{k=0}^{\infty} 2ka_k x^k + (\lambda - 1)\sum_{k=0}^{\infty} a_k x^k = 0.$$

Equating powers of x we see a power series solution must satisfy

$$(k+2)(k+1)a_{k+2} = (2k+1-\lambda)a_k \quad \text{for each } k$$

and a_0 and a_1 can be chosen arbitrarily.

(c) A calculation using the recursion relation from (b) shows that up to a multiplicative constant $w(x) = 16x^4 - 48x^2 + 12$.
