Linköpings universitet
Matematiska institutionen
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Utbildningskod: TATA27
Modul: TEN1

TATA27 Partial Differential Equations

Take-home examination 2021-05-28, 8.00–12.00

Rules, short version.

- Aids are permitted, but no collaboration with other persons is allowed.

Rules, long version.

- This is an individual examination, so you are required to answer the questions on your own.
- You may ask the teacher for clarifications (email **hans.lundmark@liu.se**). Except for that, it is not allowed to communicate in any way with other persons regarding the solutions of the problems during the exam. So you may not get help from others, and it is also not allowed to *give* help to other students who are taking this exam, for example by letting them look at your solutions.
- You can use any aids (books, computers, etc.), but you are expected to present your solutions with as much detail as if calculating by hand (like on a usual exam without aids). It is fine to consult old information from online forums, but you may not post any new questions during the exam, nor make use of questions or answers posted by others during the exam. **Cite your sources** in an appropriate way, especially if you are using "outside" sources (i.e., not the course materials). Avoid quoting text verbatim; it is much preferred if you use your own formulations.
- The solutions should be handwritten (unless you have a special permit from LiU's disability coordinator to write on a computer). Writing by hand on a tablet is fine, but please use dark text on a white background.
- You may write your answers in English or in Swedish (or some mixture thereof).

You will find the problems **on the next page**.

Each problem will be marked *pass* (3 or 2 points) or *fail* (1 or 0 points). For grade $n \in \{3, 4, 5\}$ you need at least n passed problems and at least $3n - 1$ points.

Solutions will be posted on the course webpage afterwards. Good luck!

TATA27 2021-05-28, 8.00–12.00

1. Let $\alpha > 0$ and $\beta > 0$. Determine the function $u(x,t)$ which is continuous on $[0,\pi] \times [0,\infty)$ and solves the initial-boundary value problem

$$\begin{aligned}
&u_t + \alpha u = \beta u_{xx}, \qquad 0 < x < \pi, \qquad t > 0,\\
&u_x(0,t) = u_x(\pi,t) = 0, \qquad t > 0,\\
&u(x,0) = 2\sin^2 x, \qquad 0 < x < \pi.
\end{aligned}$$

2. Choose any non-constant continuous function $\varphi(x)$ that you like. Then, with your choice of φ, let $u(x,t)$ be the weak solution (given by d'Alembert's formula) to the wave equation $u_{tt} = u_{xx}$ with initial conditions

$$\begin{aligned}
u(x,0) &= \varphi(x), \quad x \in \mathbf{R},\\
u_t(x,0) &= \begin{cases} 1, & -1 \le x \le 1,\\ 0, & \text{otherwise.} \end{cases}
\end{aligned}$$

Draw the graphs of $u(x,0)$, $u(x,1)$ and $u(x,2)$. In other words, draw the shape of the wave at times $t=0$, $t=1$ and $t=2$.

3. Consider the heat equation $u_t = u_{xx}$ on the interval $0 < x < 1$, with boundary conditions $u(0,t) = u(1,t) = 0$ for $t > 0$ and initial data $u(x,0) = 27x(1-x)^2$. What numerical approximation of $u(\frac{2}{3},1)$ would you obtain from the Crank–Nicolson finite difference scheme with $h = \frac{1}{3}$ and $\tau = 1$? (Here $h = \delta x$ and $\tau = \delta t$ are the distances between grid points.) Is this value a reasonable approximation?

4. Use the method of characteristics to solve the PDE $(1+x^2)u_x + xy\,u_y = u$ with the condition $u(0,y) = y^2$, $y \in \mathbf{R}$. What would happen if the condition instead were $u(x,0) = x^2$, $x \in \mathbf{R}$?

5. Let $\Omega \subset \mathbf{R}^n$ be open and bounded, and suppose $u \in C(\overline{\Omega}) \cap C^2(\Omega)$ satisfies

$$\Delta u \ge u^3 \text{ on } \Omega, \qquad u \le 0 \text{ on } \partial\Omega.$$

Show that $u \le 0$ on Ω.

Hint: Assume that the set $V = \{\mathbf{x} \in \Omega : u(\mathbf{x}) > 0\}$ is nonempty, and apply the weak maximum principle for subharmonic functions on V to derive a contradiction.

6. Determine the function $u(x,y)$ which is harmonic in the unit disk and takes the boundary values

$$u(x,y) = x^4 + y^4, \qquad \text{when } x^2 + y^2 = 1.$$

(For partial credit, if you are unable to find the whole function $u(x,y)$, at least determine the value $u(0,0)$ at the center of the disk.)

Hint: You can make use of Poisson's formula in its "unsummed" form, i.e., the Fourier-type series obtained by separation of variables in polar coordinates. But give your answer in the Cartesian coordinates (x,y).

Solutions for TATA27 2021-05-28

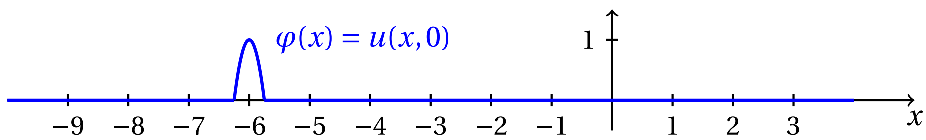
1. One can seek separated solutions $u(x,t) = X(x)\,T(t)$ directly, which leads to $X_n(x) = \cos(nx)$ and $T_n(t) = e^{-(\alpha+\beta n^2)t}$ with integers $n \geq 0$, but it is perhaps faster to let $v(x,t) = e^{\alpha t}\,u(x,t)$ to obtain the heat equation $v_t = \beta v_{xx}$ with boundary conditions $v_x(0,t) = v_x(\pi,t) = 0$ and initial condition $v(x,0) = e^0 \cdot 2\sin^2 x = 1 - \cos(2x)$, which immediately leads to the solution $v(x,t) = 1 - \cos(2x)e^{-4\beta t}$.

   Answer. $u(x,t) = e^{-at}\big(1 - \cos(2x)e^{-4\beta t}\big)$.

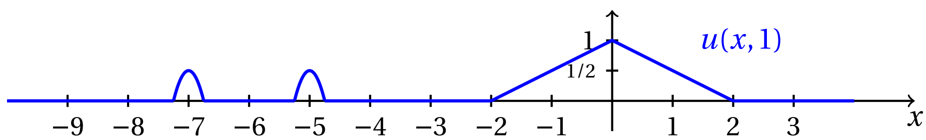
2. According to d'Alembert, the solution is

$$u(x,t) = \frac{\varphi(x-t) + \varphi(x+t)}{2} + \frac{1}{2}\int_{x-t}^{x+t} u_t(y,0)\,dy,$$

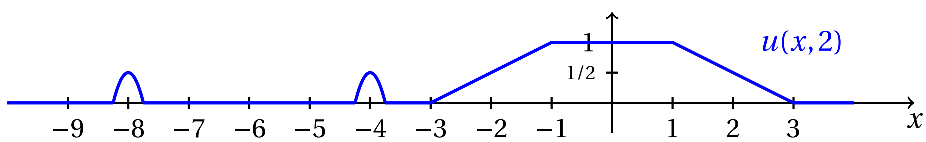
   where the integral – with our specific initial condition for u_t – equals the length of the intersection between the intervals $[x-t, x+t]$ and $[-1,1]$. The appearance of the graphs will course depend on your choice of the function φ. If you want to make your life simple, a good choice is a rather narrow "blob" with compact support, located a fair distance away from the interval $[-1,1]$, so that the two contributions $\frac{1}{2}\varphi(x-t)$ and $\frac{1}{2}\varphi(x+t)$ from $u(x,0)$ will have separated completely already by the time $t=1$, and the contributions from $u(x,0)$ and $u_t(x,0)$ don't interfere with each other until after $t=2$. For example like this (which is of course also the shape of the wave at time $t=0$):



   The wave at time $t=1$ looks as follows, since the interval $[x-1, x+1]$ starts overlapping $[-1,1]$ when $x=-2$, with the length of the intersection increasing at a constant rate as x increases from -2 to 0, and then decreasing in a symmetric way for $0 < x < 2$:



   And at time $t=2$ it's like this, with similar reasoning about the interval $[x-2, x+2]$:



3. If A and B are the apprimations to $u(\frac{1}{3},1)$ and $u(\frac{2}{3},1)$, respectively, then the first step of the Crank–Nicolson scheme with initial data $u(\frac{1}{3},0) = 27\cdot\frac{1}{3}\cdot(\frac{2}{3})^2 = 4$ and $u(\frac{2}{3},0) = 27\cdot\frac{2}{3}\cdot(\frac{1}{3})^2 = 2$ is

$$\frac{A-4}{1} = \frac{1}{2}\left(\frac{0-2\cdot 4+2}{(1/3)^2} + \frac{0-2A+B}{(1/3)^2}\right),$$
$$\frac{B-2}{1} = \frac{1}{2}\left(\frac{4-2\cdot 2+0}{(1/3)^2} + \frac{A-2B+0}{(1/3)^2}\right),$$

which simplifies to

$$\begin{pmatrix} 20 & -9 \\ -9 & 20 \end{pmatrix}\begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} -46 \\ 4 \end{pmatrix},$$

with the solution

$$\begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} 20 & -9 \\ -9 & 20 \end{pmatrix}^{-1}\begin{pmatrix} -46 \\ 4 \end{pmatrix} = \frac{1}{20^2-9^2}\begin{pmatrix} 20 & 9 \\ 9 & 20 \end{pmatrix}\begin{pmatrix} -46 \\ 4 \end{pmatrix} = \frac{1}{319}\begin{pmatrix} -884 \\ -334 \end{pmatrix}.$$

Answer. The approximate value obtained for $u(\frac{2}{3},1)$ is $-334/319$, which is completely unreasonable. The true solution $u(x,t)$ can never be negative, since it attains its minimum (and maximum) on the boundary of the domain $[-1,1]\times[0,\infty)$, where its prescribed values are nonnegative. (The problem here is of course that the time step τ is too large.)

4. The ODEs for the characteristic curve starting at the point $(x,y) = (0,s)$ are $\dot{x} = 1+x^2$ and $\dot{y} = xy$ with $x(0) = 0$ and $y(0) = s$. Thus $x(t) = \tan t$ (for $|t| < \pi/2$), and then either $y(t) \equiv 0$ or $\frac{d}{dt}\ln|y(t)| = x(t) = \tan t = -\frac{d}{dt}\ln|\cos t|$, the latter case giving $\ln|y(t)| = -\ln|\cos t| + C$ and hence $y(t) = \pm e^C/\cos t$. With the initial condition $y(0) = s$, the result is $y(t) = s/\cos t$ (both for $s = 0$ and for $s \neq 0$).

Inverting the equations $(x,y) = (\tan t, s/\cos t)$ (where $|t| < \pi/2$) gives $(t,s) = (\arctan x, y/\sqrt{1+x^2})$, so the characteristic curve through any particular point (x,y) passes through the point $(0, y/\sqrt{1+x^2})$ on the y-axis (where the values for u are given).

The evolution of $z(t) = u(x(t),y(t))$ such a characteristic curve is $\dot{z} = z$, with $z(0) = s^2$, leading to $z(t) = s^2 e^t$.

Answer. With $u(0,y) = y^2$, the solution is

$$u(x,y) = \frac{y^2}{1+x^2}\,e^{\arctan x}.$$

With $u(x,0) = x^2$, there is no solution, since the x-axis is a characteristic curve. More precisely, the given data contradict the relation that must hold along that curve according to the PDE, namely $(1+x^2)u_x(x,0) = u(x,0)$.

5. As suggested in the hint, assume that V is nonempty. For $\mathbf{x} \in V$, we have $u(\mathbf{x}) > 0$ (by the definition of V), and hence $\Delta u(\mathbf{x}) \geq u(\mathbf{x})^3 > 0$, so that u is subharmonic on V (which is a bounded set, of course, since it's a subset of the bounded set Ω, and open, since u is continuous). According to the weak maximum principle for subharmonic functions, the maximum of u on $\overline{V}$ is attained on the boundary:

$$\max_{\mathbf{x}\in\overline{V}} u(\mathbf{x}) = \max_{\mathbf{y}\in\partial V} u(\mathbf{y}).$$

There are two possibilities if $\mathbf{y} \in \partial V$: either $\mathbf{y} \in \partial\Omega$, in which case $u(\mathbf{y}) \leq 0$ by assumption, or else $\mathbf{y} \in \Omega$, in which case $u(\mathbf{y}) = 0$ since $\mathbf{y}$ lies on the boundary separating the regions in Ω where $u > 0$ and $u \leq 0$. So the right-hand side in the equality above is non-positive:

$$\max_{\mathbf{y}\in\partial V} u(\mathbf{y}) \leq 0.$$

But on the other hand, the left-hand side is obviously positive (by the definition of V):

$$\max_{\mathbf{x}\in\overline{V}} u(\mathbf{x}) > 0.$$

This contradiction implies that V must be empty, as was to be shown.

6. The given boundary values on the unit circle can be written as

$$\begin{aligned} u(\cos\varphi, \sin\varphi) &= \cos^4\varphi + \sin^4\varphi = (\cos^2\varphi + \sin^2\varphi)^2 - 2\cos^2\varphi\,\sin^2\varphi \\ &= 1 - \tfrac{1}{2}\sin^2 2\varphi = 1 - \tfrac{1}{2}\cdot\tfrac{1}{2}(1-\cos 4\varphi) = \tfrac{3}{4} + \tfrac{1}{4}\cos 4\varphi, \end{aligned}$$

so the average of u over the boundary is $3/4$, and according to the mean value property of harmonic functions, that's also the value at the center: $u(0,0) = 3/4$.

From the separation of variables leading to Poisson's formula for the disk, we know that the radial factor which goes together with the angular factor $\cos n\varphi$ is r^n, so we immediately get the formula for u in polar coordinates:

$$u(r\cos\varphi, r\sin\varphi) = \tfrac{3}{4}r^0 + \tfrac{1}{4}r^4\cos 4\varphi.$$

To convert this to Cartesian coordinates, we can use parts of the above calculation in reverse:

$$\begin{aligned} u &= \tfrac{3}{4} + r^4\cdot\tfrac{1}{4}\cos 4\varphi \\ &= \tfrac{3}{4} + r^4\big(1 - 2\cos^2\varphi\,\sin^2\varphi - \tfrac{3}{4}\big) \\ &= \tfrac{3}{4} + \tfrac{1}{4}(r^2)^2 - 2(r\cos\varphi)^2\,(r\sin\varphi)^2 \\ &= \tfrac{3}{4} + \tfrac{1}{4}(x^2+y^2)^2 - 2x^2y^2 \\ &= \tfrac{3}{4} + \tfrac{1}{4}(x^4 - 6x^2y^2 + y^4). \end{aligned}$$

Answer. $u(x,y) = \frac{1}{4}(3 + x^4 - 6x^2y^2 + y^4)$. Or, equivalently, $u(x,y) = x^4 + y^4 + \frac{3}{4}\big(1 - (x^2+y^2)^2\big)$, making it more obvious that u really satisfies the given boundary condition.