Linköpings universitet
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Utbildningskod: TATA27
Modul: TEN1

# TATA27 Partiella differentialekvationer

## Tentamen 2021-08-16 kl. 14.00–18.00

No aids allowed. You may write your answers in English or in Swedish.

Each problem is marked *pass* (3 or 2 points) or *fail* (1 or 0 points). For grade $n \in \{3,4,5\}$ you need at least $n$ passed problems and at least $3n-1$ points.

Solutions will be posted on the course webpage afterwards. Good luck!

1. Solve the following initial-boundary value problem for the heat equation:

$$\begin{aligned} &u_t = u_{xx}, \qquad 0 < x < \pi/2, \qquad t > 0, \\ &u(0,t) = 1 \quad \text{and} \quad u_x(\pi/2,t) = 0, \qquad t > 0, \\ &u(x,0) = 1 - \sin^3 x, \qquad 0 < x < \pi/2. \end{aligned}$$

2. Solve the wave equation $u_{tt} = u_{xx}$ on the half-line $x > 0$ with boundary condition $u(0,t) = 0$ for $t > 0$ and initial conditions $u(x,0) = xe^{-x}$ and $u_t(x,0) = 0$ for $x > 0$.

3. Use the method of characteristics to find $u(x,y)$ such that $xu_x + u_y = u$ and $u(x,0) = h(x)$, where $h\colon \mathbf{R} \to \mathbf{R}$ is a given (differentiable) function.

4. Prove the weak maximum principle for harmonic functions: If $\Omega$ is a bounded open set in $\mathbf{R}^n$, and if $u$ is continuous on $\overline{\Omega}$ and harmonic on $\Omega$, then the maximum of $u$ on $\overline{\Omega}$ is attained on the boundary $\partial\Omega$.

   (Hint: Recall that it's useful to consider the function $v(\mathbf{x}) = u(\mathbf{x}) + \varepsilon\,|\mathbf{x}|^2$.)

5. There is also a weak maximum principle for the heat equation, say $u_t = u_{xx}$ on the domain $(x,t) \in [0,1] \times [0,T]$. What does it say? If the solution is approximated using the explicit finite difference scheme which is based on the discretizations

$$u_t(x,t) \approx \frac{u(x,t+\tau) - u(x,t)}{\tau}, \quad u_{xx}(x,t) \approx \frac{u(x+h,t) - 2u(x,t) + u(x-h,t)}{h^2},$$

   then does the numerical solution satisfy a corresponding (discrete) maximum principle?

6. Determine a weak solution (for $t \ge 0$) of the inviscid Burgers equation $u_t + uu_x = 0$, with the initial condition

$$u(x,0) = \begin{cases} 1, & x < 0, \\ x, & x > 0. \end{cases}$$

   Make sure that the Rankine–Hugoniot condition is satisfied: the velocity of the shock equals the average of the left and right limits of $u$ there.

   (Here it is understood that the weak formulation is "the usual one", based upon interpreting the PDE as the conservation law $u_t + (\tfrac{1}{2}u^2)_x = 0$.)

## Solutions for TATA27 2021-08-16

1. The function $v(x,t) = u(x,t) - 1$ satisfies $v_t = v_{xx}$ with homogeneous boundary conditions $v(0,t) = v_x(\pi/2,t) = 0$. The basic separated solutions in this situation are $v_n(x,t) = e^{-n^2 t} \sin nx$ for $n = 1,3,5,\ldots$, and since the initial condition for $v$ is

$$v(x,0) = -\sin^3 x = -\tfrac{3}{4}\sin x + \tfrac{1}{4}\sin 3x = -\tfrac{3}{4}v_1(x,0) + \tfrac{1}{4}v_3(x,0),$$

we immediately obtain the solution as $u = 1 + v$, where

$$v(x,t) = -\tfrac{3}{4}v_1(x,t) + \tfrac{1}{4}v_3(x,t) = -\tfrac{3}{4}e^{-t}\sin x + \tfrac{1}{4}e^{-9t}\sin 3x.$$

**Answer.** $u(x,t) = 1 - \tfrac{3}{4}e^{-t}\sin x + \tfrac{1}{4}e^{-9t}\sin 3x$.

2. The solution on the half-line is given by the usual d'Alembert formula, if we extend the initial data to **odd** functions on the whole real line: $u(x,0) = xe^{-|x|}$ and $u_t(x,0) = 0$ for $x \in \mathbf{R}$.

**Answer.** $u(x,t) = \tfrac{1}{2}(x+t)e^{-|x+t|} + \tfrac{1}{2}(x-t)e^{-|x-t|}$.

3. For a fixed $s \in \mathbf{R}$, the characteristic curve $(x(t), y(t))$ through the point $(s,0)$ is given by $\dot{x} = x$, $x(0) = s$ and $\dot{y} = 1$, $y(0) = 0$, hence $x(t) = se^t$ and $y(t) = t$. Along that curve, $z(t) = u(x(t), y(t))$ satisfies $\dot{z} = z$ with $z(0) = h(s)$, so that $z(t) = h(s)e^t$. With $t = y$ and $s = xe^{-t} = xe^{-y}$, this gives $u = h(s)e^t = h(xe^{-y})e^y$.

**Answer.** $u(x,y) = h(xe^{-y})e^y$.

4. See the course materials.

5. The maximum of $u$ on the rectangle $[0,1] \times [0,T]$ in the $(x,t)$-plane is attained on the parabolic boundary, i.e., the union of the lower edge (where $t = 0$) and the vertical edges (where $x = 0$ or $x = 1$). (This is physically plausible, as we don't expect heat conduction to give rise to "hot spots" in the interior of the domain.) The numerical solution satisfies a discrete maximum principle, provided that the parameter $s = \tau/h^2$ equals at most $1/2$; see the course materials.

6. If the shock curve is $x = \xi(t)$, then for $t \geq 0$ the solution has the form

$$u(x,t) = \begin{cases} u_L(x,t), & x < \xi(t), \\ u_R(x,t), & x > \xi(t), \end{cases}$$

where $u_L = 1$ and $u_R = x/(1+t)$ are found by the method of characteristics. The shock curve is then determined by the Rankine–Hugoniot condition

$$\xi'(t) = \frac{u_L + u_R}{2}\bigg|_{x=\xi(t)} = \frac{1}{2}\left(1 + \frac{\xi(t)}{1+t}\right)$$

together with the initial condition $\xi(0) = 0$. This is a linear ODE for $\xi(t)$, which can be solved via multiplication by the integrating factor $1/\sqrt{1+t}$:

$$\frac{d}{dt}\left(\frac{\xi(t)}{\sqrt{1+t}}\right) = \frac{1}{2\sqrt{1+t}}, \quad \text{etc.}$$

**Answer.** The solution is

$$u(x,t) = \begin{cases} 1, & x < \xi(t), \\ x/(1+t), & x > \xi(t), \end{cases} \quad \text{where} \quad \xi(t) = 1 + t - \sqrt{1+t}.$$