Linköpings universitet
Matematiska institutionen
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Utbildningskod: TATA27
Modul: TEN1

TATA27 Partiella differentialekvationer

Tentamen 2023-05-30 kl. 8.00–12.00

No aids allowed (except drawing tools, such as rulers, of course). You may write your answers in English or in Swedish, or some mixture thereof.

Each problem is marked *pass* (3 or 2 points) or *fail* (1 or 0 points). For grade $n \in \{3,4,5\}$ you need at least n passed problems and at least $3n-1$ points.

Solutions will be posted on the course webpage afterwards. Good luck!

1. Solve the wave equation $u_{tt} = c^2 u_{xx}$ for $0 < x < \pi/2$ and $t > 0$, with the boundary conditions $u_x(0,t) = 0$ and $u(\pi/2,t) = 0$ and the initial conditions $u(x,0) = 2\cos(3x)$ and $u_t(x,0) = 7\cos(5x)$.

2. Formulate the weak and strong maximum principles for harmonic functions on a nonempty open set Ω in $\mathbf{R}^n$.

3. Use the method of characteristics to solve the PDE $xu_x + (1+y^2)u_y = u$ with the condition $u(x,0) = f(x)$, where $f \in C^1(\mathbf{R})$ is some given function.

4. Suppose $u(x,y,z)$ is harmonic on the open ball $x^2+y^2+z^2 < 4$ and continuous out to the boundary, with the boundary values given by $u(x,y,z) = \exp(z)$ for $x^2+y^2+z^2 = 4$. Determine the value $u(0,0,0)$.

5. Consider the PDE $u_t + cu_x = \alpha u_{xx}$ where α and c are positive constants.
   (a) Suggest (with motivation) some physical situation which may be modelled by this equation.
   (b) Find a change of variables of the form $(\tau,\xi) = (t, f(x,t))$ which reduces the PDE to the heat equation $u_\tau = \alpha u_{\xi\xi}$.

6. Consider the following finite element approach to the heat equation $u_t = u_{xx}$ on the interval $0 < x < 1$, with Dirichlet boundary conditions $u = 0$ at the endpoints: introduce N nodes x_k such that $0 < x_1 < x_2 < \cdots < x_N < 1$, and seek an approximate solution of the form $u(x,t) = \sum_{k=1}^{N} c_k(t)\,\varphi_k(x)$, where $\varphi_k(x)$ is the standard “tent-shaped” basis function which is piecewise linear, equals 1 at the node x_k, and equals 0 at all other nodes.
   (a) Derive a semi-weak formulation suitable for the FEM approach above. (Multiply by a test function $\varphi(x)$ and integrate by parts to move a derivative from u_{xx} to φ.)
   (b) Show that this gives an ODE system of the form $A\frac{d\mathbf{c}}{dt} + B\mathbf{c} = \mathbf{0}$ for the vector $\mathbf{c}(t) = \big(c_1(t),\dots,c_N(t)\big)^T$, where A and B are symmetric tridiagonal $N \times N$ matrices with positive entries on the main diagonal.

Solutions for TATA27 2023-05-30

1. The basic separated solutions that satisfy the PDE and the boundary conditions are $u(x,t) = \cos(\omega x)\cos(\omega c t)$ and $u(x,t) = \cos(\omega x)\sin(\omega c t)$ with ω an odd positive integer, and the general solution is a Fourier-type series, a linear combination of these infinitely many basic solutions. However, the particular initial conditions given here are satisfied by a very simple linear combination, where only two terms (with $\omega = 3$ and $\omega = 5$) are nonzero.

   Answer. $u(x,t) = 2\cos(3x)\cos(3ct) + \frac{7}{5c}\cos(5x)\sin(5ct)$.

2. The weak maximum principle says that if u is harmonic on a **bounded** nonempty open set Ω and $u \in C(\overline{\Omega})$, then the maximum and mimimum of u on $\overline{\Omega}$ (which exist by the extreme value theorem) are attained on the boundary $\partial\Omega$. The strong maximum principle says that if Ω moreover is **connected**, then the maximum and mimimum are attained *only* on the boundary, not in the interior, unless u is constant on $\overline{\Omega}$.

   [A related statement which is also sometimes called the strong maximum principle is that if u is harmonic on a connected (but not necessarily bounded) open set Ω and has a local maximum or minimum at some point in Ω, then u is constant on Ω.]

3. For a fixed $s \in \mathbf{R}$, the characteristic curve $(x(t), y(t))$ through the point $(s, 0)$ is given by $\dot{x} = x$, $x(0) = s$ and $\dot{y} = 1 + y^2$, $y(0) = 0$, hence $x(t) = se^t$ and $y(t) = \tan t$, $|t| < \pi/2$. Along that curve, $z(t) = u(x(t), y(t))$ satisfies $\dot{z} = z$ with $z(0) = f(s)$, so that $z(t) = f(s)\,e^t$. With $t = \arctan y$ and $s = x/e^t = xe^{-\arctan y}$, this gives the solution $u = f(s)\,e^t = f(xe^{-\arctan y})\,e^{\arctan y}$ (defined in the whole plane).

   Answer. $u(x,t) = f(xe^{-\arctan y})\,e^{\arctan y}$, $(x,y) \in \mathbf{R}^2$.

4. The mean value property for harmonic functions says that $u(0,0,0)$ equals the average of the values on the boundary sphere $x^2 + y^2 + z^2 = 4$. To compute this average, we parametrize the sphere with spherical coordinates

   $$(x,y,z) = (r\cos\varphi\sin\theta, r\sin\varphi\sin\theta, r\cos\theta)$$

   with $r = 2$, which gives $dS = r \cdot r\sin\theta \cdot d\theta\,d\varphi = 4\sin\theta\,d\theta\,d\varphi$ (using the scale factors r and $r\sin\theta$ for the θ and φ directions). This gives the mean value integral

   $$u(0,0,0) = \fint_S u\,dS = \frac{\int_S u\,dS}{\int_S dS} = \frac{\int_{\theta=0}^{\pi}\left(\int_{\varphi=0}^{2\pi} \exp(2\cos\theta)\,4\sin\theta\,d\varphi\right)d\theta}{\int_{\theta=0}^{\pi}\left(\int_{\varphi=0}^{2\pi} 4\sin\theta\,d\varphi\right)d\theta}$$

   $$= \frac{8\pi\int_{\theta=0}^{\pi}\exp(2\cos\theta)\,\sin\theta\,d\theta}{8\pi\int_{\theta=0}^{\pi}\sin\theta\,d\theta} = \frac{\left[-\frac{1}{2}\exp(2\cos\theta)\right]_{\theta=0}^{\pi}}{\left[-\cos\theta\right]_{\theta=0}^{\pi}}$$

   $$= \frac{-\frac{1}{2}\big(\exp(-2) - \exp(2)\big)}{2} = \tfrac{1}{2}\sinh(2).$$

5. (a) This is a diffusion–advection equation, describing for example diffusion of some chemical (of concentration u) in a fluid flowing with constant speed c in a long tube along the x-axis. It's a conservation law $u_t + J_x = 0$, where the flux term $J = -\alpha u_x + cu$ combines Fick's law of diffusion $J_1 = -\alpha u_x$ with an advection term $J_2 = cu$.

   (b) Inspired by part (a), we "go with the flow" and let $\xi = x - ct$ (together with $\tau = t$). A short computation with the chain rule verifies that we do get the heat (or diffusion) equation in these new variables.

6. (a) With a test function $\varphi(x)$ which is zero at the endpoints $x = 0$ and $x = 1$, we get

$$
\begin{aligned}
0 &= \int_0^1 (u_t - u_{xx})\,\varphi\,dx = \int_0^1 u_t\,\varphi\,dx - \int_0^1 u_{xx}\,\varphi\,dx \\
&= \int_0^1 u_t\,\varphi\,dx - \Bigl(\underbrace{\bigl[u_x\,\varphi\bigr]_0^1}_{=0} - \int_0^1 u_x\,\varphi_x\,dx\Bigr) \\
&= \int_0^1 (u_t\,\varphi + u_x\,\varphi_x)\,dx,
\end{aligned}
$$

so a suitable semi-weak formulation is to require this last integral to be zero for all test functions $\varphi(x)$ such that $\varphi(0) = \varphi(1) = 0$.

   (b) The FEM approximation to the solution is obtained by taking $u(x,t) = \sum_{k=1}^{N} c_k(t)\,\varphi_k(x)$ and requiring the integral above to be zero when $\varphi = \varphi_m$ (for $1 \le m \le N$). Writing dot for $\frac{d}{dt}$ and prime for $\frac{d}{dx}$ we get

$$
\begin{aligned}
0 &= \int_0^1 \left(\Bigl(\sum_{k=1}^{N} \dot c_k(t)\,\varphi_k(x)\Bigr)\varphi_m(x) + \Bigl(\sum_{k=1}^{N} c_k(t)\,\varphi_k'(x)\Bigr)\varphi_m'(x) \right) dx \\
&= \sum_{k=1}^{N} \Biggl(\underbrace{\Bigl(\int_0^1 \varphi_k(x)\,\varphi_m(x)\,dx\Bigr)}_{=A_{mk}} \dot c_k(t) + \underbrace{\Bigl(\int_0^1 \varphi_k'(x)\,\varphi_m'(x)\,dx\Bigr)}_{=B_{mk}} c_k(t) \Biggr) \\
&= \sum_{k=1}^{N} \bigl(A_{mk}\dot c_k(t) + B_{mk} c_k(t) \bigr).
\end{aligned}
$$

This sum is entry number m in the column vector $A\frac{d\mathbf{c}}{dt} + B\mathbf{c}$, so the whole vector must be equal to the zero vector, as was to be shown. It's obvious from the definitions that the matrices A and B satisfy $A_{mk} = A_{km}$ and $B_{mk} = B_{km}$, and they are tridiagonal since if the indices k and m are more than one step apart, then φ_k and φ_m have disjoint supports, so that $\varphi_k(x)\,\varphi_m(x) = 0$ and $\varphi_k'(x)\,\varphi_m'(x) = 0$ for all $x \in [0,1]$. Also, it's clear that $A_{kk} = \int_0^1 \varphi_k(x)^2\,dx > 0$ and $B_{kk} = \int_0^1 \varphi_k'(x)^2\,dx > 0$ for $1 \le k \le N$.