Linköpings universitet
Matematiska institutionen
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Utbildningskod: TATA27
Modul: TEN1

TATA27 Partiella differentialekvationer

Tentamen 2023-08-14 kl. 14.00–18.00

No aids allowed (except drawing tools, such as rulers, of course). You may write your answers in English or in Swedish, or some mixture thereof.

Each problem is marked *pass* (3 or 2 points) or *fail* (1 or 0 points). For grade $n \in \{3, 4, 5\}$ you need at least n passed problems and at least $3n - 1$ points.

Solutions will be posted on the course webpage afterwards. Good luck!

1. Solve the heat equation $u_t = u_{xx}$ for $0 < x < \pi$ and $t > 0$, with the initial condition $u(x, 0) = x(\pi - x)$ and the Dirichlet boundary conditions $u(0, t) = u(\pi, t) = 0$. (Hint: The solution takes the form of a Fourier-type series.)

2. Determine all functions $f(r)$, $r > 0$, such that $u(x, y) = f\big(\sqrt{x^2 + y^2}\big)$ is harmonic on $\mathbf{R}^2 \setminus \{(0, 0)\}$.

3. Consider the wave equation $u_{tt} = \frac{1}{4} u_{xx}$ for $x \in \mathbf{R}$ and $t > 0$, with initial conditions

$$u(x, 0) = \begin{cases} \sin x, & \text{for } 0 < x < \pi, \\ 0, & \text{otherwise,} \end{cases} \qquad u_t(x, 0) = 0 \quad \text{for } x \in \mathbf{R}.$$

   Let $u(x, t)$ be the (weak) solution given by d'Alembert's formula. Draw the graph of the function $x \mapsto u(x, 1)$. (In other words, draw the shape of the wave after one unit of time.)

4. Use the method of characteristics to find $u(x, y)$ such that $x^2 u_x + u_y = u$ and $u(x, 0) = \sin x$. In what region of the plane is the problem's solution u determined?

5. Poisson's integral formula for the unit ball in $\mathbf{R}^n$ says that if u is harmonic on the open ball $B(\mathbf{0}, 1)$ and continuous on the closed ball $\overline{B(\mathbf{0}, 1)}$, then

$$u(\mathbf{a}) = \fint_{S(\mathbf{0},1)} \frac{1 - |\mathbf{a}|^2}{|\mathbf{x} - \mathbf{a}|^n}\, u(\mathbf{x})\, dS(\mathbf{x}) \qquad \text{for } \mathbf{a} = B(\mathbf{0}, 1),$$

   where $S(\mathbf{0}, 1)$ is the unit sphere, and the symbol $\fint$ denotes the mean value integral. Use this to derive Poisson's integral formula for a ball $B(\mathbf{0}, r)$ in $\mathbf{R}^n$ of arbitrary radius $r > 0$.

6. Formulate the weak maximum principle for the heat equation $u_t = \Delta u$, where $t \in (0, T)$ and $\mathbf{x} \in \Omega$ (a bounded domain in $\mathbf{R}^n$). Use this to show uniqueness of solutions to the initial-boundary value problem where the initial temperature $u(\mathbf{x}, 0)$ and the boundary values $u(\mathbf{x}, t)$, $\mathbf{x} \in \partial\Omega$, are prescribed.

Solutions for TATA27 2023-08-14

1. The basic separated solutions satisfying the PDE and the boundary conditions are $u_n(x,t) = \sin(nx)\, e^{-n^2 t}$ for integers $n \ge 1$, and the sought solution takes the form $u(x,t) = \sum_{n=1}^{\infty} c_n u_n(x,t)$. In order to satisfy the initial condition, we need $x(\pi - x) = u(x,0) = \sum_{n=1}^{\infty} c_n u_n(x,0) = \sum_{n=1}^{\infty} c_n \sin(nx)$ for $0 < x < \pi$. We multiply this by $\sin(kx)$ and integrate from 0 to π, using that $\int_0^{\pi} \sin(nx)\sin(kx)\,dx = 0$ for $k \neq n$, to obtain

$$c_k \underbrace{\int_0^{\pi} \sin^2(kx)\,dx}_{=\pi/2} = \int_0^{\pi} x(\pi - x)\sin(kx)\,dx,$$

so that

$$\begin{aligned} c_k &= \frac{2}{\pi}\int_0^{\pi} x(\pi - x)\sin(kx)\,dx \\ &= \frac{2}{\pi}\left[(\pi x - x^2)\frac{-\cos(kx)}{k} - (\pi - 2x)\frac{-\sin(kx)}{k^2} + (-2)\frac{\cos(kx)}{k^3}\right]_0^{\pi} \\ &= \frac{4\bigl(1 - (-1)^k\bigr)}{\pi k^3}. \end{aligned}$$

Answer.

$$u(x,t) = \sum_{n=1}^{\infty} \frac{4\bigl(1-(-1)^n\bigr)}{\pi n^3} \sin(nx)\, e^{-n^2 t},$$

or equivalently, since only odd-numbered $c_n = c_{2m+1}$ are nonzero,

$$u(x,t) = \sum_{m=0}^{\infty} \frac{8}{\pi(2m+1)^3} \sin\bigl((2m+1)x\bigr)\, e^{-(2m+1)^2 t}.$$

2. If you remember the formula for the Laplacian in polar coordinates, $\Delta u = u_{rr} + \frac{1}{r^2} u_{\varphi\varphi} + \frac{1}{r} u_r$, you can use that to obtain $f''(r) + \frac{1}{r} f'(r) = 0$. Otherwise, just compute $u_x(x,y) = f'\bigl(\sqrt{x^2+y^2}\bigr)\cdot x/\sqrt{x^2+y^2}$, and so on, to derive that same ODE. Multiplication by the integrating factor r gives $(r f'(r))' = 0$, and after two integrations we find $f(r) = A \ln r + B$ with arbitrary constants A and B.

   Answer. $f(r) = A\ln r + B$, $r > 0$.

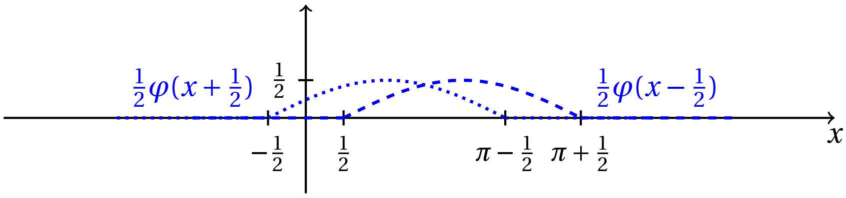
3. Since $c = \sqrt{1/4} = 1/2$, d'Alembert's formula says that

$$u(x,t) = \frac{\varphi(x - \frac{1}{2}t) + \varphi(x + \frac{1}{2}t)}{2},$$

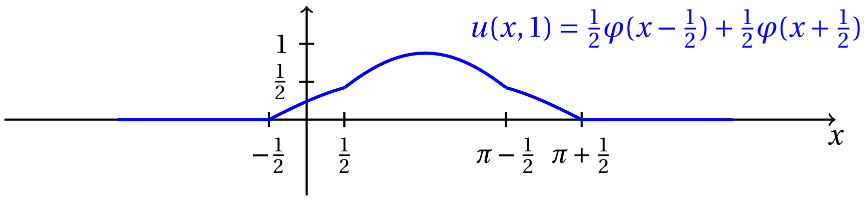
where $\varphi(x) = u(x,0)$, and in particular

$$u(x,1) = \frac{\varphi(x - \frac{1}{2}) + \varphi(x + \frac{1}{2})}{2}.$$

To draw the graph, we can first plot the two terms $\frac{1}{2}\varphi(x \pm \frac{1}{2})$ separately:

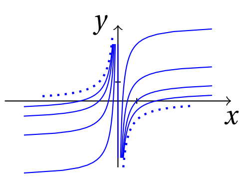


Then adding them up (graphically) gives the answer:



4. For a fixed $s \in \mathbf{R}$, the characteristic curve $(x(t), y(t))$ through the point $(s,0)$ is given by $\dot{x} = x^2$ and $\dot{y} = 1$, with initial conditions $x(0) = s$ and $y(0) = 0$. Hence, $x(t) = s/(1 - st)$ and $y(t) = t$. (The solution for $x(t)$ is valid for all $t \in \mathbf{R}$ if $s = 0$, but only for $t < 1/s$ if $s > 0$ and for $t > 1/s$ if $s < 0$.) Along such a curve, $z(t) = u(x(t), y(t))$ satisfies $\dot{z} = z$ with $z(0) = \sin(s)$, so that $z(t) = \sin(s)\, e^t$. With $t = y$ and $s = x/(1 + xt) = x/(1 + xy)$, this gives $u = \sin(s)\, e^t = \sin(x/(1 + xy))\, e^y$. The characteristics computed above, emanating from points on the x-axis, fill out the region $xy > -1$ (between the two dotted curves in the figure below), so this is the region where the solution u is determined by the given conditions. If we approach either of the dotted curves from inside this region, for example along a line $x = \text{const.}$, the solution is seen to oscillate wildly, so it cannot be continued up to (or across) the boundary in a nice way.

Answer. $u(x,y) = \sin\left(\dfrac{x}{1+xy}\right) e^y$, for $xy > -1$.



5. Let A_n be the $(n-1)$-dimensional surface area of the unit sphere $S(\mathbf{0},1)$ in R^n. Suppose u is harmonic on $B(\mathbf{0},r)$ and continuous out to the boundary. Then $v(\mathbf{x}) = u(r\mathbf{x})$ is harmonic on the unit ball $B(\mathbf{0},1)$ and continuous out to the boundary. (Rescaling a harmonic function like this doesn't destroy harmonicity; this can be shown by calculation using the chain rule, or argued more abstractly, for example by noticing that the rescaled function still has the mean value property.) Thus, for $\mathbf{a} \in B(\mathbf{0},r)$, we can apply the formula in the problem to the rescaled function v, with $\mathbf{b} = \frac{1}{r}\mathbf{a} \in B(\mathbf{0},1)$ playing the role of $\mathbf{a}$, and writing $\mathbf{y}$ instead of $\mathbf{x}$, to obtain the value

$$\begin{aligned} u(\mathbf{a}) = v(\tfrac{1}{r}\mathbf{a}) = v(\mathbf{b}) &= \fint_{S(\mathbf{0},1)} \frac{1-|\mathbf{b}|^2}{|\mathbf{y}-\mathbf{b}|^n}\, v(\mathbf{y})\, dS(\mathbf{y}) \\ &= \fint_{S(\mathbf{0},1)} \frac{1-\left|\frac{1}{r}\mathbf{a}\right|^2}{\left|\mathbf{y}-\frac{1}{r}\mathbf{a}\right|^n}\, v(\mathbf{y})\, dS(\mathbf{y}) = \frac{1}{A_n}\int_{S(\mathbf{0},1)} \frac{\frac{1}{r^2}}{\frac{1}{r^n}} \frac{r^2-|\mathbf{a}|^2}{|r\mathbf{y}-\mathbf{a}|^n}\, u(r\mathbf{y})\, dS(\mathbf{y}). \end{aligned}$$

Under the change of variables $\mathbf{x} = r\mathbf{y}$, the surface area element scales as $dS(\mathbf{x}) = r^{n-1}\, dS(\mathbf{y})$, and the new region of integration is the sphere $S(\mathbf{0},r)$, whose surface area is $A_n r^{n-1}$, so we get

$$\begin{aligned} u(\mathbf{a}) &= \frac{1}{A_n}\int_{S(\mathbf{0},r)} \frac{r^n}{r^2}\frac{r^2-|\mathbf{a}|^2}{|\mathbf{x}-\mathbf{a}|^n}\, u(\mathbf{x})\, \frac{dS(\mathbf{x})}{r^{n-1}} \\ &= \frac{1}{A_n r^{n-1}}\int_{S(\mathbf{0},r)} \frac{r^{n-2}\left(r^2-|\mathbf{a}|^2\right)}{|\mathbf{x}-\mathbf{a}|^n}\, u(\mathbf{x})\, dS(\mathbf{x}) \\ &= \fint_{S(\mathbf{0},r)} \frac{r^{n-2}\left(r^2-|\mathbf{a}|^2\right)}{|\mathbf{x}-\mathbf{a}|^n}\, u(\mathbf{x})\, dS(\mathbf{x}), \end{aligned}$$

which is the sought formula.

6. Suppose $u(\mathbf{x},t)$ solves the heat equation on the set $\Omega \times (0,T)$ and is continuous out to the boundary. Then the weak maximum principle says that the maximum value of u on the compact set $\overline{\Omega} \times [0,T]$ is attained at a point on the parabolic boundary of $\overline{\Omega} \times [0,T]$, i.e., at a point in that set such that either $t = 0$ or $\mathbf{x} \in \partial\Omega$. (And applying this to the function $-u$ shows that the minimum of u is likewise attained on the parabolic boundary.)

   Now if u_1 and u_2 are two solutions to the same initial-boundary value problem, then their difference $u = u_1 - u_2$ is a solution to the heat equation which is zero on the parabolic boundary, and hence by the weak maximum (and minimum) principle, it must be zero throughout $\overline{\Omega} \times [0,T]$. So $u_1 = u_2$, which means that the initial-boundary value problem can have at most one solution.