Linköpings universitet
Matematiska institutionen
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Utbildningskod: TATA27
Modul: TEN1

TATA27 Partiella differentialekvationer

Tentamen 2024-01-05 kl. 8.00–12.00

No aids allowed (except drawing tools, such as rulers, of course). You may write your answers in English or in Swedish, or some mixture thereof.

Each problem is marked *pass* (3 or 2 points) or *fail* (1 or 0 points). For grade $n \in \{3, 4, 5\}$ you need at least n passed problems and at least $3n - 1$ points.

Solutions will be posted on the course webpage afterwards. Good luck!

1. Use the method of characteristics to find a function $u(x, y)$ which satisfies $u_x - 2xyu_y = 0$ and $u(0, y) = f(y)$, where $f\colon \mathbf{R} \to \mathbf{R}$ is some given differentiable function.

2. Find a solution $u(x, t)$ to the heat equation $u_t = u_{xx}$ for $x \in \mathbf{R}$ and $t > 0$, with initial condition $u(x, 0) = e^{-x^2}$.

   (Hint: Remember the fundamental solution $S(x, t) = \frac{1}{\sqrt{4\pi t}} \exp\left(-\frac{x^2}{4t}\right)$. Or use that the Fourier transform of $e^{-\alpha x^2}$ is $\int_{\mathbf{R}} e^{-\alpha x^2} e^{-i\xi x}\,dx = \sqrt{\frac{\pi}{\alpha}} \exp\left(-\frac{\xi^2}{4\alpha}\right)$ for $\alpha > 0$.)

3. (a) Formulate the weak maximum principle for harmonic functions.

   (b) Let $B = \{\mathbf{x} \in \mathbf{R}^n : |\mathbf{x}| < 1\}$ be the open unit ball in $\mathbf{R}^n$. Assume that the functions $u(\mathbf{x})$ and $|\mathbf{x}|^2\, u(\mathbf{x})$ are harmonic on B and continuous on $\overline{B}$. Show that u is identically zero on $\overline{B}$.

4. Determine the solution $u(x, t)$ (in the form of a Fourier-type series) to the wave equation $u_{tt} = u_{xx}$ for $0 < x < 1$ and $t > 0$ with the boundary conditions $\frac{\partial u}{\partial x}(0, t) = 0 = u(1, t)$ for $t > 0$ and the initial conditions $u(x, 0) = 0$ and $\frac{\partial u}{\partial t}(x, 0) = 1 - x$ for $0 < x < 1$.

5. Formulate what it means for $u(x, t)$ to be a weak solution (on the whole space $\mathbf{R}^2$) to the advection equation $u_t + cu_x = 0$, and prove that $u(x, t) = f(x - ct)$ is a weak solution for any locally integrable function $f\colon \mathbf{R} \to \mathbf{R}$.

6. Show that the PDE $u_{xx} = e^{2x} u_{yy}$ is hyperbolic in the whole plane $\mathbf{R}^2$. Find characteristic coordinates $(r, s) = (r(x, y), s(x, y))$ and express the PDE in terms of them.

Solutions for TATA27 2024-01-05

1. For a fixed $s \in \mathbf{R}$, the characteristic curve $(x(t), y(t))$ through the point $(0, s)$ is given by $\dot{x} = 1$ and $\dot{y} = -2xy$ with initial conditions $x(0) = 0$ and $y(0) = s$. We immediately get $x(t) = t$. This gives $\dot{y} + 2ty = 0$, and using the integrating factor e^{t^2} we find that $y(t) = se^{-t^2}$. The PDE implies that u is constant along each such curve, and the additional condition says that this constant value is $f(s)$. And from $(x, y) = (t, se^{-t^2}) \iff (t, s) = (x, ye^{x^2})$ we see that $s = ye^{x^2}$ is the value of the parameter s which picks out the characteristic curve which passes through a given point (x, y).

   Answer. $u(x, y) = f\big(ye^{x^2}\big)$, for $(x, y) \in \mathbf{R}^2$.

2. The function $S(x, t)$ satisfies the heat equation in the region $t > 0$, and since the coefficients of the heat equation don't depend on t, a translation of a solution in the t direction is still a solution. Also, since the heat equation is linear, a constant times a solution is still a solution. Thus, the function

   $$u(x, t) = \sqrt{\pi} S(x, t + \tfrac{1}{4}) = \frac{1}{\sqrt{4(t + \frac{1}{4})}} \exp\left(-\frac{x^2}{4(t + \frac{1}{4})}\right)$$

   satisfies the heat equation $u_t = u_{xx}$ in the region $t > -\frac{1}{4}$, and it clearly also satisfies the initial condition $u(x, 0) = e^{-x^2}$.

   Alternatively, the same solution can be found as follows. If $U(\xi, t)$ is the Fourier transform of $u(x, t)$ with respect to x, then for each fixed ξ it satisfies the ODE $U_t = -\xi^2 U$ (the Fourier transform of the heat equation $u_t = u_{xx}$) with the initial value $U(\xi, 0) = \mathscr{F}[e^{-x^2}] = \sqrt{\pi} e^{-\xi^2/4}$ (where we used the given Fourier transform formula with $\alpha = 1$). Solving this ODE gives

   $$U(\xi, t) = U(\xi, 0)\, e^{-\xi^2 t} = \sqrt{\pi} e^{-\xi^2/4} e^{-\xi^2 t} = \sqrt{\pi} e^{-\xi^2(4t+1)/4}.$$

   Taking the inverse Fourier transform of this (using the same formula but in reverse and with $\alpha = 1/(4t + 1)$ instead) gives

   $$u(x, t) = \frac{1}{\sqrt{4t + 1}} \exp\left(-\frac{x^2}{4t + 1}\right),$$

   which agrees with what we obtained above.

   Answer. $u(x, t) = \dfrac{1}{\sqrt{4t + 1}} \exp\left(-\dfrac{x^2}{4t + 1}\right)$.

3. (a) Suppose that Ω is a **bounded** nonempty open set in $\mathbf{R}^n$, and that u is harmonic on Ω and continuous on the closure $\overline{\Omega}$. Then the maximum and minimum of u on $\overline{\Omega}$ are assumed on the boundary $\partial\Omega$.

   (b) Let $v(\mathbf{x}) = u(\mathbf{x}) - |\mathbf{x}|^2 u(\mathbf{x}) = (1-|\mathbf{x}|^2)u(\mathbf{x})$. According to the assumptions, v is harmonic on B and continuous on $\overline{B}$, and moreover it is zero on the boundary sphere $S = \partial B$, since $|\mathbf{x}| = 1$ there. Since B is a bounded domain, the maximum and minimum values of the harmonic function v on the closed ball $\overline{B}$ are attained on the boundary S, so these values are both zero, which forces v to be identically zero on $\overline{B}$. This means that u is identically zero on B, since the factor $1-|\mathbf{x}|^2$ is nonzero there and hence can be cancelled from the equality $0 = v(\mathbf{x}) = (1-|\mathbf{x}|^2)u(\mathbf{x})$. Since u is continuous out to the boundary (by assumption), it must also be zero on $\overline{B}$, as was to be shown.

4. The separated solutions $u(x,t) = X(x)T(t)$ matching the given boundary conditions are

$$\cos\big((n+\tfrac{1}{2})\pi x\big)\cos\big((n+\tfrac{1}{2})\pi t\big) \quad \text{and} \quad \cos\big((n+\tfrac{1}{2})\pi x\big)\sin\big((n+\tfrac{1}{2})\pi t\big),$$

where $n \geq 0$ is an integer. Since $u(x,0) = 0$, only terms of the second kind appear in the solution, and thus we need to determine $(c_n)_{n=0}^{\infty}$ such that

$$u(x,t) = \sum_{n=0}^{\infty} c_n \cos\big((n+\tfrac{1}{2})\pi x\big)\sin\big((n+\tfrac{1}{2})\pi t\big)$$

satisfies the other initial condition $u_t(x,0) = 1-x$, i.e.,

$$\sum_{n=0}^{\infty} c_n (n+\tfrac{1}{2})\pi \cos\big((n+\tfrac{1}{2})\pi x\big) = 1-x, \qquad 0<x<1.$$

We multiply this by $\cos\big((k+\tfrac{1}{2})\pi x\big)$ and integrate from $x=0$ to $x=1$. Due to the orthogonality of the functions $\cos\big((n+\tfrac{1}{2})\pi x\big)$ only the term with $n=k$ survives on the left-hand side, so we get (after some calculation)

$$c_k(k+\tfrac{1}{2})\pi \underbrace{\int_0^1 \cos^2\big((k+\tfrac{1}{2})\pi x\big)\,dx}_{=\cdots=\frac{1}{2}} = \underbrace{\int_0^1 (1-x)\cos\big((k+\tfrac{1}{2})\pi x\big)\,dx}_{=\cdots=\frac{1}{(k+\frac{1}{2})^2\pi^2}},$$

i.e., $c_k = 2(k+\tfrac{1}{2})^{-3}\pi^{-3}$ for $k \in \mathbf{N}$.

Answer. $u(x,t) = \displaystyle\sum_{n=0}^{\infty} \frac{2}{(n+\frac{1}{2})^3\pi^3} \cos\big((n+\tfrac{1}{2})\pi x\big)\sin\big((n+\tfrac{1}{2})\pi t\big).$

5. By definition, u is a weak solution iff $\iint_{\mathbf{R}^2} u(\varphi_t + c\varphi_x)\,dx\,dt = 0$ for all test functions $\varphi(x,t)$ (i.e., C^∞-functions with compact support).

   To show that this holds when $u(x,t) = f(x-ct)$, let φ be an arbitrary test function and consider the integral

$$\iint_{\mathbf{R}^2} f(x-ct)\big(\varphi_t(x,t) + c\varphi_x(x,t)\big)\,dx\,dt.$$

   The change of variables $(y,s) = (x-ct, t)$, with Jacobian determinant equal to 1 so that $dy\,ds = dx\,dt$, turns this integral into

$$\iint_{\mathbf{R}^2} f(y)\big(\varphi_t(y+cs,s) + c\varphi_x(y+cs,s)\big)\,dy\,ds.$$

   Here the expression in brackets equals $\frac{\partial}{\partial s}\Big(\varphi(y+cs,s)\Big)$ by the chain rule, so we can integrate with respect to s first, to obtain

$$\int_{y=-\infty}^{\infty} f(y)\,\underbrace{\Big[\varphi(y+cs,s)\Big]_{s=-\infty}^{\infty}}_{=0 \text{ since } \varphi \text{ has cpt supp.}}\,dy = 0,$$

   as desired.

6. The PDE is of the form $A(x,y)u_{xx} + 2B(x,y)u_{xy} + C(x,y)u_{yy} = 0$, with $A = 1$, $B = 0$ and $C = -e^{2x}$, so that $AC - B^2 = -e^{2x}$ is negative in the whole plane $\mathbf{R}^2$; this shows that the PDE is hyperbolic everywhere.

   At each point (x,y), the quadratic form $Q(v_1,v_2) = A(x,y)v_1^2 + 2B(x,y)v_1v_2 + C(x,y)v_2^2 = v_1^2 - e^{2x}v_2^2 = (v_1 - e^x v_2)(v_1 + e^x v_2)$ is indefinite, with $Q = 0$ when the vector (v_1,v_2) is parallel to $(e^x,1) = \nabla(e^x+y)$ or to $(e^x,-1) = \nabla(e^x-y)$. This shows that $(r,s) = (e^x+y, e^x-y)$ are characteristic coordinates. (Since $e^x = \frac{1}{2}(r+s)$ and $y = \frac{1}{2}(r-s)$ we see that they are defined in the region $r+s>0$ in the rs-plane.)

   The chain rule gives $u_x = e^x(u_r+u_s)$, $u_{xx} = e^x(u_r+u_s) + e^{2x}(u_{rr} + 2u_{rs} + u_{ss})$, $u_y = u_r - u_s$ and $u_{yy} = u_{rr} - 2u_{rs} + u_{ss}$, so the PDE becomes $0 = u_{xx} - e^{2x}u_{yy} = e^x(u_r+u_s) + e^{2x}(u_{rr}+2u_{rs}+u_{ss}) - e^{2x}(u_{rr}-2u_{rs}+u_{ss}) = e^x(u_r + u_s) + 4e^{2x}u_{rs}$, or equivalently $0 = u_{rs} + \frac{1}{4e^x}(u_r+u_s) = u_{rs} + \frac{1}{4(r+s)}(u_r+u_s)$.

   Answer. $(r,s) = (e^x+y, e^x-y)$ are characteristic coordinates, in terms of which the PDE becomes $u_{rs} + \dfrac{u_r+u_s}{4(r+s)} = 0$ (for $r+s>0$).