Elementära Maclaurinutvecklingar

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Maclaurins formel

$$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \ldots + \frac{f^{(n)}(0)}{n!}x^n + \mathcal{O}(x^{n+1}).$$

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$$\begin{aligned}
e^x &= 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \ldots + \frac{x^n}{n!} + \mathcal{O}(x^{n+1}),\\
\sin x &= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots + (-1)^{n-1}\frac{x^{2n-1}}{(2n-1)!} + \mathcal{O}(x^{2n+1}),\\
\cos x &= 1 - \frac{x^2}{2} + \frac{x^4}{4!} - \ldots + (-1)^{n}\frac{x^{2n}}{(2n)!} + \mathcal{O}(x^{2n+2}),\\
\ln(1+x) &= x - \frac{x^2}{2} + \frac{x^3}{3} - \ldots + (-1)^{n-1}\frac{x^{n}}{n} + \mathcal{O}(x^{n+1}),\\
\arctan x &= x - \frac{x^3}{3} + \frac{x^5}{5} - \ldots + (-1)^{n-1}\frac{x^{2n-1}}{2n-1} + \mathcal{O}(x^{2n+1}),\\
(1+x)^\alpha &= 1 + \alpha x + \binom{\alpha}{2}x^2 + \binom{\alpha}{3}x^3 + \ldots + \binom{\alpha}{n}x^n + \mathcal{O}(x^{n+1}),
\end{aligned}$$

där

$$\binom{\alpha}{2} = \frac{\alpha(\alpha-1)}{2}, \quad \binom{\alpha}{3} = \frac{\alpha(\alpha-1)(\alpha-2)}{3!}, \ldots$$

Vi har att $f(x) = \mathrm{e}^x$ uppfyller $f(x) = f'(x) = \ldots = f^{(n)}(x) = \mathrm{e}^x$.

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$$e^x = f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \ldots + \frac{f^{(n)}(0)}{n!}x^n + \mathcal{O}(x^{n+1})$$

Vi har att $f(x) = \mathrm{e}^x$ uppfyller $f(x) = f'(x) = \ldots = f^{(n)}(x) = \mathrm{e}^x$.
Så $f(0) = f'(0) = \ldots = f^{(n)}(0) = 1$.

$$\begin{aligned}
\mathrm{e}^x &= f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \ldots + \frac{f^{(n)}(0)}{n!}x^n + \mathcal{O}(x^{n+1}) \\
&= 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \ldots + \frac{x^n}{n!} + \mathcal{O}(x^{n+1}).
\end{aligned}$$

sin x

$f(x) = \sin x$ uppfyller $f'(x) = \cos x$, $f''(x) = -\sin x$, $f^{(3)}(x) = -\cos x$, $f^{(4)}(x) = \sin x \ldots$

$\sin x$

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Detta ger

$$f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f^{(3)}(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4 + \dots$$
$$= 0 + 1x + 0x^2 - \frac{x^3}{3!} + 0x^4 + \dots$$

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Alltså

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots + (-1)^{n-1}\frac{x^{2n-1}}{(2n-1)!} + \mathcal{O}(x^{2n+1}).$$