

TATA43, Kompletterande problem, neuronnät

K1. Bestäm funktionalmatriserna av $\bar{f} : \mathbb{R}^2 \rightarrow \mathbb{R}^2$, $\bar{g} : \mathbb{R}^6 \rightarrow \mathbb{R}^2$ och $\bar{h} : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ som ges av

$$\bar{f}(\bar{x}) = \begin{pmatrix} f_1(x_1, x_2) \\ f_2(x_1, x_2) \end{pmatrix} = \begin{pmatrix} x_1 + 2x_2 + 5 \\ 3x_1 + 4x_2 + 6 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 5 \\ 6 \end{pmatrix}$$

$$\bar{g}(\bar{w}) = \begin{pmatrix} g_1(w_1, w_2, w_3, w_4, w_5, w_6) \\ g_2(w_1, w_2, w_3, w_4, w_5, w_6) \end{pmatrix} = \begin{pmatrix} w_1 & w_2 \\ w_3 & w_4 \end{pmatrix} \begin{pmatrix} 7 \\ 8 \end{pmatrix} + \begin{pmatrix} w_5 \\ w_6 \end{pmatrix}$$

$$\bar{h}(\bar{t}) = \begin{pmatrix} h_1(t_1, t_2, t_3) \\ h_2(t_1, t_2, t_3) \\ h_3(t_1, t_2, t_3) \end{pmatrix} = \begin{pmatrix} e^{t_1} + t_3^2 \\ \sin t_2 \\ \ln t_3 \end{pmatrix}$$

K2. Antag att

$$\bar{t} = \begin{pmatrix} t_1 \\ t_2 \\ t_3 \end{pmatrix} = \bar{f}(\bar{x}, w_1, w_2, w_3, w_4, w_5, w_6) = \begin{pmatrix} f_1(x_1, x_2, w_1, w_2, w_3, w_4, w_5, w_6) \\ f_2(x_1, x_2, w_1, w_2, w_3, w_4, w_5, w_6) \\ f_3(x_1, x_2, w_1, w_2, w_3, w_4, w_5, w_6) \end{pmatrix} = \begin{pmatrix} w_1 & w_2 \\ w_3 & w_4 \\ w_5 & w_6 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$\bar{u} = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = \bar{\sigma}(\bar{t}) = \begin{pmatrix} \sigma(t_1) \\ \sigma(t_2) \\ \sigma(t_3) \end{pmatrix}, \text{ d\u00e4r } \sigma(t) = \max(t, 0) = (t)_+$$

$$y = g(\bar{u}, w_7, w_8, w_9, w_{10}) = g(u_1, u_2, u_3, w_7, w_8, w_9, w_{10}) = (w_7, w_8, w_9) \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} + w_{10}$$

a) Ange de fem funktionalmatriserna $\frac{\partial(\bar{t})}{\partial(\bar{x})} = \frac{\partial(t_1, t_2, t_3)}{\partial(x_1, x_2)}$,

$$\frac{\partial(\bar{t})}{\partial(w_1, w_2, w_3, w_4, w_5, w_6)} = \frac{\partial(t_1, t_2, t_3)}{\partial(w_1, w_2, w_3, w_4, w_5, w_6)}, \quad \frac{\partial(\bar{u})}{\partial(\bar{t})} = \frac{\partial(u_1, u_2, u_3)}{\partial(t_1, t_2, t_3)},$$

$$\frac{\partial(y)}{\partial(\bar{u})} = \frac{\partial(y)}{\partial(u_1, u_2, u_3)} \quad \text{och} \quad \frac{\partial(y)}{\partial(w_7, w_8, w_9, w_{10})}$$

b) Om $\begin{pmatrix} w_1 & w_2 \\ w_3 & w_4 \\ w_5 & w_6 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ -3 & 1 \\ 2 & 3 \end{pmatrix}$, $(w_7, w_8, w_9) = (-1, 2, 1)$, $w_{10} = 3$ och $\bar{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$,

beräkna \bar{t} , \bar{u} och y , samt de fem funktionalmatriserna ovan.

c) Med värdena i b), använd kedjeregeln f\u00f6r att ber\u00e4kna funktionalmatrisen

$$\frac{\partial(y)}{\partial(w_1, w_2, w_3, w_4, w_5, w_6)} \text{ samt ange } \frac{\partial(y)}{\partial(\bar{w})} = \frac{\partial(y)}{\partial(w_1, w_2, w_3, w_4, w_5, w_6, w_7, w_8, w_9, w_{10})}$$

K3. Vi ska ta ett (eller två) steg med gradientmetoden för att träna upp ett litet neuronät $y = f(\bar{x})$. Antag att det finns två träningspunkter, $\tilde{x} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$ ska ge $\tilde{y} = -2$ och $\hat{x} = \begin{pmatrix} 3 \\ -1 \end{pmatrix}$ ska ge $\hat{y} = 2$. Neuronätet har två lager, $f(\bar{x}) = f_2(\bar{f}_1(\bar{x}))$, där $\bar{f}_1(\bar{x}) = \bar{\sigma}(A_1\bar{x})$ med $A_1 = \begin{pmatrix} w_1 & w_2 \\ w_3 & w_4 \end{pmatrix}$ och $\bar{\sigma}\left(\begin{pmatrix} t_1 \\ t_2 \end{pmatrix}\right) = \begin{pmatrix} (t_1)_+ \\ (t_2)_+ \end{pmatrix} = \begin{pmatrix} t_1 \\ t_2 \end{pmatrix}_+$, $f_2(\bar{u}) = A_2\bar{u}$ med $A_2 = \begin{pmatrix} w_5 & w_6 \end{pmatrix}$.

Vi vill minimera $\ell(\bar{w}) = \frac{1}{2}[(f(\tilde{x}, \bar{w}) - \tilde{y})^2 + (f(\hat{x}, \bar{w}) - \hat{y})^2]$ där $\bar{w} = (w_1, w_2, w_3, w_4, w_5, w_6)$.

Som startpunkt i \mathbb{R}^6 har vi $\bar{w}_{(0)} = (1, 2, -1, 1, 1, -2)$.

Ange matriserna A_1 och A_2 i startpunkten och beräkna $\tilde{t} = A_1\tilde{x}$, $\hat{t} = A_1\hat{x}$, $\tilde{u} = \bar{\sigma}(\tilde{t})$, $\hat{u} = \bar{\sigma}(\hat{t})$, $f(\tilde{x}) = A_2\tilde{u}$ och $f(\hat{x}) = A_2\hat{u}$. Vad blir $\ell(\bar{w}_{(0)})$?

Beräkna $\nabla\ell(\bar{w}_{(0)})$ och bestäm en ny punkt $\bar{w}_{(1)} = \bar{w}_{(0)} - \eta\nabla\ell(\bar{w}_{(0)})$ för $\eta = 0.1$.

I punkten $\bar{w}_{(1)}$, ange \tilde{t} , \hat{t} , \tilde{u} , \hat{u} , $f(\tilde{x})$, $f(\hat{x})$, $\ell(\bar{w}_{(1)})$ och $\nabla\ell(\bar{w}_{(1)})$.

Extra: Ta steget till nästa punkt $\bar{w}_{(2)}$ och beräkna $\ell(\bar{w}_{(2)})$. Jämför resultaten för $\eta = 0.1$ och $\eta = 0.05$ i detta steg.

Svar / lösningar

$$\mathbf{K1.} \quad \bar{f}'(\bar{x}) = \frac{\partial(\bar{f})}{\partial(\bar{x})} = \frac{\partial(f_1, f_2)}{\partial(x_1, x_2)} = \begin{pmatrix} (f_1)'_{x_1} & (f_1)'_{x_2} \\ (f_2)'_{x_1} & (f_2)'_{x_2} \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$$

$$\begin{pmatrix} g_1(w_1, w_2, w_3, w_4, w_5, w_6) \\ g_2(w_1, w_2, w_3, w_4, w_5, w_6) \end{pmatrix} = \begin{pmatrix} 7w_1 + 8w_2 + w_5 \\ 7w_3 + 8w_4 + w_6 \end{pmatrix} \Rightarrow$$

$$\frac{\partial(\bar{g})}{\partial(\bar{w})} = \frac{\partial(g_1, g_2)}{\partial(w_1, w_2, w_3, w_4, w_5, w_6)} = \begin{pmatrix} 7 & 8 & 0 & 0 & 1 & 0 \\ 0 & 0 & 7 & 8 & 0 & 1 \end{pmatrix}$$

$$\frac{\partial(\bar{h})}{\partial(\bar{t})} = \frac{\partial(h_1, h_2, h_3)}{\partial(t_1, t_2, t_3)} = \begin{pmatrix} e^{t_1} & 0 & 2t_3 \\ 0 & \cos t_2 & 0 \\ 0 & 0 & 1/t_3 \end{pmatrix}$$

$$\mathbf{K2.} \text{ a) } \bar{t} = \begin{pmatrix} t_1 \\ t_2 \\ t_3 \end{pmatrix} = \begin{pmatrix} w_1x_1 + w_2x_2 \\ w_3x_1 + w_4x_2 \\ w_5x_1 + w_6x_2 \end{pmatrix} \Rightarrow \frac{\partial(\bar{t})}{\partial(\bar{x})} = \frac{\partial(t_1, t_2, t_3)}{\partial(x_1, x_2)} = \begin{pmatrix} w_1 & w_2 \\ w_3 & w_4 \\ w_5 & w_6 \end{pmatrix} \text{ och}$$

$$\frac{\partial(\bar{t})}{\partial(w_1, w_2, w_3, w_4, w_5, w_6)} = \frac{\partial(t_1, t_2, t_3)}{\partial(w_1, w_2, w_3, w_4, w_5, w_6)} = \begin{pmatrix} x_1 & x_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & x_1 & x_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & x_1 & x_2 \end{pmatrix}$$

$$\frac{\partial(\bar{u})}{\partial(\bar{t})} = \frac{\partial(u_1, u_2, u_3)}{\partial(t_1, t_2, t_3)} = \begin{pmatrix} H(t_1) & 0 & 0 \\ 0 & H(t_2) & 0 \\ 0 & 0 & H(t_3) \end{pmatrix}, \text{ där } H(t) = \begin{cases} 1, & t > 0 \\ 0, & t < 0 \end{cases}$$

$$y = w_7u_1 + w_8u_2 + w_9u_3 + w_{10} \Rightarrow \frac{\partial(y)}{\partial(\bar{u})} = \frac{\partial(y)}{\partial(u_1, u_2, u_3)} = (w_7, w_8, w_9) \text{ och}$$

$$\frac{\partial(y)}{\partial(w_7, w_8, w_9, w_{10})} = (u_1, u_2, u_3, 1)$$

$$\text{b) } \bar{x} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \Rightarrow \bar{t} = \begin{pmatrix} 1 & 2 \\ -3 & 1 \\ 2 & 3 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 4 \\ -5 \\ 7 \end{pmatrix} \Rightarrow \bar{u} = \begin{pmatrix} 4 \\ 0 \\ 7 \end{pmatrix} \Rightarrow y = (-1, 2, 1) \begin{pmatrix} 4 \\ 0 \\ 7 \end{pmatrix} + 3 = 6 \Rightarrow$$

$$\frac{\partial(\bar{t})}{\partial(\bar{x})} = \begin{pmatrix} 1 & 2 \\ -3 & 1 \\ 2 & 3 \end{pmatrix}, \frac{\partial(\bar{t})}{\partial(w_1, w_2, w_3, w_4, w_5, w_6)} = \begin{pmatrix} 2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 1 \end{pmatrix}, \frac{\partial(\bar{u})}{\partial(\bar{t})} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$\frac{\partial(y)}{\partial(\bar{u})} = (-1, 2, 1) \text{ och } \frac{\partial(y)}{\partial(w_7, w_8, w_9, w_{10})} = (4, 0, 7, 1)$$

c) Kedjeregeln ger

$$\frac{\partial(y)}{\partial(w_1, w_2, w_3, w_4, w_5, w_6)} = \frac{\partial(y)}{\partial(\bar{u})} \frac{\partial(\bar{u})}{\partial(\bar{t})} \frac{\partial(\bar{t})}{\partial(w_1, w_2, w_3, w_4, w_5, w_6)} =$$

$$= (-1, 2, 1) \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 1 \end{pmatrix} = (-2, -1, 0, 0, 2, 1) \Rightarrow$$

$$\frac{\partial(y)}{\partial(\bar{w})} = \frac{\partial(y)}{\partial(w_1, w_2, w_3, w_4, w_5, w_6, w_7, w_8, w_9, w_{10})} = (-2, -1, 0, 0, 2, 1, 4, 0, 7, 1)$$

K3. För $\bar{w}_{(0)}$ fås $A_1 = \begin{pmatrix} 1 & 2 \\ -1 & 1 \end{pmatrix}$ och $A_2 = (1 \ -2) \Rightarrow \tilde{t} = A_1 \begin{pmatrix} -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \Rightarrow \tilde{u} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

$\Rightarrow f(\tilde{x}) = A_2 \begin{pmatrix} 1 \\ 2 \end{pmatrix} = -3$. Liknande fås $\hat{t} = \begin{pmatrix} 1 \\ -4 \end{pmatrix} \Rightarrow \hat{u} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \Rightarrow f(\hat{x}) = 1$.

$\ell(\bar{w}_{(0)}) = 0.5 \cdot ((-3 - (-2))^2 + (1 - 2)^2) = 0.5 \cdot (1^2 + 1^2) = 1$.

$y = A_2 \bar{u} = u_1 w_5 + u_2 w_6$ ger $\frac{\partial(y)}{\partial(w_5, w_6)} = (u_1 \ u_2) = \bar{u}^T$ och $\frac{\partial(y)}{\partial(\bar{u})} = (w_5 \ w_6) = A_2$.

$\bar{u} = (\bar{t})_+$ ger $\frac{\partial(\bar{u})}{\partial(\bar{t})} = \begin{pmatrix} H(t_1) & 0 \\ 0 & H(t_2) \end{pmatrix} = H(\bar{t})$ (där $H(t) = 1, t > 0, H(t) = 0, t < 0$).

$\bar{t} = A_1 \bar{x} = \begin{pmatrix} x_1 w_1 + x_2 w_2 \\ x_1 w_3 + x_2 w_4 \end{pmatrix}$ ger $\frac{\partial(\bar{t})}{\partial(w_1, w_2, w_3, w_4)} = \begin{pmatrix} x_1 & x_2 & 0 & 0 \\ 0 & 0 & x_1 & x_2 \end{pmatrix} = M(\bar{x})$.

Kedjeregeln $\Rightarrow \frac{\partial(y)}{\partial(w_1, w_2, w_3, w_4)} = \frac{\partial(y)}{\partial(\bar{u})} \frac{\partial(\bar{u})}{\partial(\bar{t})} \frac{\partial(\bar{t})}{\partial(w_1, w_2, w_3, w_4)} = A_2 H(\bar{t}) M(\bar{x})$.

$\nabla \ell(\bar{w}_{(0)}) = (f(\tilde{x}, \bar{w}_{(0)}) - \tilde{y}) \nabla f(\tilde{x}, \bar{w}_{(0)}) + (f(\hat{x}, \bar{w}_{(0)}) - \hat{y}) \nabla f(\hat{x}, \bar{w}_{(0)}) =$

$= (-3 - (-2)) \cdot (A_2 H(\tilde{t}) M(\tilde{x}), \tilde{u}^T) + (1 - 2) \cdot (A_2 H(\hat{t}) M(\hat{x}), \hat{u}^T) =$

$-\left((1 \ -2) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} -1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 \end{pmatrix}, 1, 2 \right) - \left((1 \ -2) \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 3 & -1 & 0 & 0 \\ 0 & 0 & 3 & -1 \end{pmatrix}, 1, 0 \right)$

$= -(-1, 1, 2, -2, 1, 2) - (3, -1, 0, 0, 1, 0) = (-2, 0, -2, 2, -2, -2)$.

$\bar{w}_{(1)} = \bar{w}_{(0)} - 0.1 \cdot (-2, 0, -2, 2, -2, -2) = (1.2, 2, -0.8, 0.8, 1.2, -1.8)$, ger nya matriser

$A_1 = \begin{pmatrix} 1.2 & 2 \\ -0.8 & 0.8 \end{pmatrix}, A_2 = (1.2 \ -1.8) \Rightarrow \tilde{t} = \begin{pmatrix} 0.8 \\ 1.6 \end{pmatrix} \Rightarrow \tilde{u} = \begin{pmatrix} 0.8 \\ 1.6 \end{pmatrix} \Rightarrow f(\tilde{x}) = -1.92,$

$\hat{t} = \begin{pmatrix} 1.6 \\ -3.2 \end{pmatrix} \Rightarrow \hat{u} = \begin{pmatrix} 1.6 \\ 0 \end{pmatrix} \Rightarrow f(\hat{x}) = 1.92$ och $\ell(\bar{w}_{(1)}) = 0.5 \cdot (0.08^2 + 0.08^2) = 0.0064$.

Obs att $\ell(\bar{w}_{(1)}) < \ell(\bar{w}_{(0)})$ så de nya matriserna är bättre anpassade till träningsdata.

$\nabla \ell(\bar{w}_{(1)}) = 0.08 \cdot (-1.2, 1.2, 1.8, -1.8, 0.8, 1.6) - 0.08 \cdot (3.6, -1.2, 0, 0, 1.6, 0)$

$= (-0.384, 0.192, 0.144, -0.144, -0.064, 0.128)$.

Extra: $\eta = 0.1$ ger $\bar{w}_{(2)} = (1.2384, 1.9808, -0.8144, 0.8144, 1.2064, -1.8128)$ och $\ell(\bar{w}_{(2)}) \approx 0.0058$. $\eta = 0.05$ ger $\bar{w}_{(2)} = (1.2192, 1.9904, -0.8072, 0.8072, 1.2032, -1.8064)$ och $\ell(\bar{w}_{(2)}) \approx 0.0001$ (mycket bättre).