Kursens mål: mest om integrationsteori

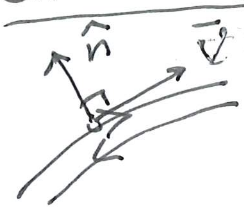
envar: varbyte ⊕ partiell integration

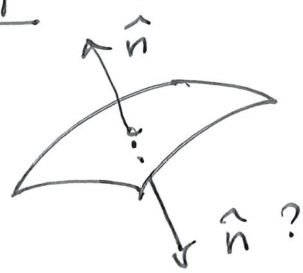
derivata: kedjeregel ⊕ Leibniz regel

flervar: varbyte ⊕ "partiell int" = Stokes formel / sats.

vektoran

Ett viktig begrepp: orientering





Yta

upåt / inåt etc

Högeror.

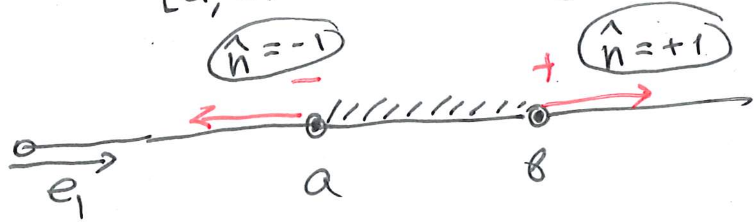
Varför:

envar

$$\int_a^b f'(x)\,dx = f(b) - f(a) \quad \longmapsto \quad \boxed{\iint_{Yta} ? = ?}$$

$$\int_{[a,b]} f'(x)\,dx \overset{?}{=} \left(+f(b)\right) + \left(-f(a)\right)$$

$$\{a, b\} = \partial[a,b] = \text{randen}$$



$$\int_{[a,b]} df(x) = \int_{\partial[a,b]} f(x)\cdot \hat{n}\, ds$$

Repetition $\mathbb{R}^1$ $\mathbb{R}^2$ $\mathbb{R}^3$

- $B \subset A$, $x \in A$, $\bar{u} \parallel \bar{v}$, $\bar{u} = \underline{e}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = (x, y, z)$
- $\bar{u} \bullet \bar{v} = x_1 x_2 + y_1 y_2 + z_1 z_2$ $\quad * \rightarrow 2 \rightarrow 3 \quad 3 \rightarrow 1 \quad 1 \rightarrow 2$
- $\bar{u} \times \bar{v} = \begin{vmatrix} \bar{e}_1 & \bar{e}_2 & \bar{e}_3 \\ x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{vmatrix} = \left(\begin{vmatrix} y_1 & z_1 \\ y_2 & z_2 \end{vmatrix}, \begin{vmatrix} z_1 & x_1 \\ z_2 & x_2 \end{vmatrix}, \begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix} \right)$
- Area $(\bar{u}, \bar{v}) = |\bar{u} \times \bar{v}|$ i $\mathbb{R}^3$
- Vol. $(\bar{u}, \bar{v}, \bar{w}) = |(u \times v) \bullet w| = |u \bullet (v \times w)| = $ etc.

  (trippelprodukt $\quad (a, b, c) = a \bullet (b \times c) = \det\begin{pmatrix} a \\ b \\ c \end{pmatrix}$

- $\bar{r} = (x, y)$ eller $\bar{r} = (x, y, z)$ ortsvektor
- $\bar{r} : \mathbb{R}^1 \rightarrow \mathbb{R}^2$ kurvor (plankurva)

  $\mathbb{R}^1 \rightarrow \mathbb{R}^3$ (rymdkurva)
- $\bar{r} : \mathbb{R}^2 \rightarrow \mathbb{R}^3$ ytor (rymdytor)

Ex: • $x = y^2$ • $x^2 + y^2 = 1$ • $\begin{cases} x = e^t \\ y = e^{-t} \end{cases}$

En yta : $S \longrightarrow$
- $z = f(x, y)$ graf
- $\bar{r} = \bar{r}(u, v)$ parameterform
- $F(x, y, z) = 0$ implicit

Ex. $F = x^2 + 3y^2 = 4 \longrightarrow x = \sqrt{4 - 3y^2}$ eller $x = -\sqrt{\ldots}$

Parameterframställning:

$\bar{r} \begin{cases} x = 2 \cos u \\ y = \frac{2}{\sqrt{3}} \sin u \end{cases} \quad 0 \leq u \leq 2\pi$

$\nabla F = (2x, 6y)_{(1,1)} = (2, 6) \rightarrow (1, 3)$

$x + 3y = 4$

Tangentlinjen i $(1, 1)$

$u = \frac{\pi}{3}$: $\cos\frac{\pi}{3} = \frac{1}{2}$

$\sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$

$\bar{r}'(\frac{\pi}{3}) = (-2 \sin u ; \frac{2}{\sqrt{3}} \cos u)$

$\underbrace{\bar{r}(\tfrac{\pi}{3}) + t\, \bar{r}'(\tfrac{\pi}{3})}_{\text{Taylor}}$

Def $\mathbb{R}^2 \to \mathbb{R}^2$ eller $\mathbb{R}^3 \to \mathbb{R}^k$ kallas vectorvärda functioner ($k \geq 2$), också vectorfält.

Deriverbara: ofta minst C^1, iblan C^2.

$\bar{F}(\bar{x}) : \mathbb{R}^k \to \mathbb{R}^n$, $1 \leq k, n \leq 3$.

Sats $\bar{F}(\bar{x})$ är deriverbar (C^k) (avbildning, fält) (d.v.s $\bar{F}(\bar{x} + h\bar{e}_k) = \bar{F}(\bar{x}) + h\bar{A}_k + \overline{o(h)}$.) $\Leftrightarrow$ varje koordinat är deriverbar (C^k).

Ex. • $\bar{F}(x,y,z) = (yz, xz, yx) \in C^k$ – ett fält

• $\bar{F}(t) = (\cos t, \sin t, e^t)$ – en rymdkurva

• $\bar{F}(u,v) = (u, 2v, u-v)$ – en rymdyta.

Låt $A(u,v,..)$ bli en vectorvärdfunktion

δ vara $\left\{ \frac{\partial}{\partial u}, \frac{\partial}{\partial v}, d ; \frac{d}{du} \text{ etc.} \right\}$

1) $\delta(\bar{A} \times \bar{B}) = \delta(\bar{A}) \times \bar{B} + \bar{A} \times \delta(\bar{B})$

2) $\delta(\bar{A} \cdot \bar{B}) = \delta(\bar{A}) \cdot \bar{B} + \ldots$

3) $\delta(\varphi \cdot \bar{A}) = \delta(\varphi) \cdot \bar{A} + \varphi \cdot \delta(\bar{A})$

"skalärfält

Tripel produkter: a) $\bar{A} \cdot (\bar{B} \times \bar{C}) = \det \begin{pmatrix} A \\ B \\ C \end{pmatrix} = \text{Vol}$

b) $A \times (B \times C) = (C \cdot A) B - (A \cdot B) C$

$\begin{cases} 1 \times (1 \times 2) = -2 \\ 2 \times (1 \times 2) = 1 \end{cases}$

$\delta(A \times (B \times C)) =$ en i taget!

Differentialen

$$d\varphi(x,y) = \varphi'_x\,dx + \varphi'_y\,dy \qquad \mathbb{R}^2,\ \mathbb{R}^3$$

$$\Delta\varphi \approx d\varphi$$

Ex1. $d(xe^{yz}) = e^{yz}dx + xze^{yz}dy + xye^{yz}dz$

Ex.2 $\sqrt{\frac{1,2}{0,9}} = \sqrt{\frac{x}{y}}$ vid $(1,1)$ med $\Delta\bar{r} = (0,2,\,-0,1)$

$$\sqrt{\frac{x}{y}} \approx \sqrt{\frac{1}{1}} + \frac{1}{2\sqrt{xy}}\,dx - \frac{\sqrt{x}}{2\sqrt{y^3}}\,dy = 1 + \frac{1}{2}\cdot 0,2 + \frac{1}{2}\,0,1 =$$

$= 1,15$ (närmvärde) $\quad = 1,1547\ldots$

Gradienten

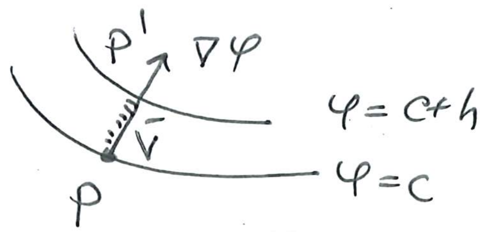
$d\varphi = \nabla\varphi \cdot d\bar{r}$ där $\nabla\varphi = (\varphi'_x, \varphi'_y, \varphi'_z) = \text{grad}\,\varphi$

- $\nabla\varphi$ är en vectorfält (potentiellt)
- φ är en potential.
- $\nabla\varphi$ pekar i den riktning i vilken φ växer snabbast
- den maximala hastigheten $= |\nabla\varphi(P)|$ i P.
- riktningsderivata

$$\frac{\partial\varphi}{\partial\bar{v}} = \nabla\varphi\cdot\bar{v}$$

- $\nabla\varphi(P) \perp$ nivåmängden i P.
- det vinkelräta avståndet vid P mellan de närbelägna nivåytorna $\varphi = c$ och $\varphi = c + h$ $(h \sim 0)$ är

$$\Delta s \approx \frac{h}{|\nabla\varphi(P)|}$$



$h = \varphi(P') - \varphi(P) = \Delta\varphi = \nabla\varphi(P)\cdot\bar{v}$

$|\bar{v}| = \Delta s \qquad \nabla\varphi(P) \parallel \bar{v}$

Observera att

$F = \nabla\varphi = (\varphi'_x, \varphi'_y, \varphi'_z)$ uppfyller

$$\underbrace{\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z} = \frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x} = \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}}_{\text{Rotation}} = 0$$

I ett allmänt fall

$$\operatorname{rot}\overline{F} = \nabla \times \overline{F} = \begin{vmatrix} e_1 & e_2 & e_3 \\ \partial_x & \partial_y & \partial_z \\ F_1 & F_2 & F_3 \end{vmatrix} \begin{matrix} \\ \leftarrow \overline{\nabla} \\ \leftarrow \overline{F} \end{matrix}$$

kryssprod.

nabla

Divergens

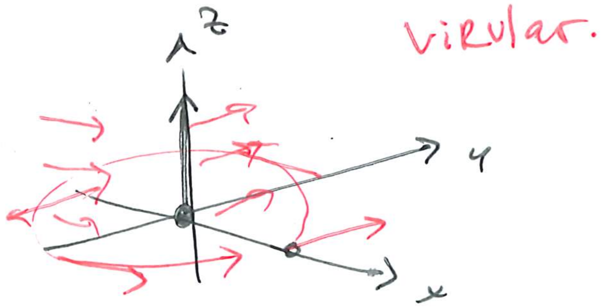
$$\nabla \cdot \overline{F} = (\partial_x, \partial_y, \partial_z) \cdot (F_1, F_2, F_3) = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$$

Ex. $\overline{F} = (x, y, z)$ $\qquad \overline{A} = (-y, x, 0)$

$\operatorname{div}\overline{F} = 3$ $\qquad \operatorname{div}\overline{A} = 0$

$$\operatorname{rot}\overline{F} = \begin{vmatrix} i & j & k \\ \partial_x & \partial_y & \partial_z \\ x & y & z \end{vmatrix} = (0, 0, 0)$$

$$\operatorname{rot}\overline{A} = \begin{vmatrix} i & j & k \\ \cdot & \cdot & \cdot \\ -y & x & 0 \end{vmatrix} = (0; 0; 2) = 2\overline{e}_3$$



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