Fö 3 Gauss sats. Divergensen. Singulära vektorfält

(1)

Repetition: $\bar{F}$ är ett C^1-vektorfält i $D \subset \mathbb{R}^3$

$$\operatorname{div} \bar{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} = \nabla \cdot \bar{F}$$

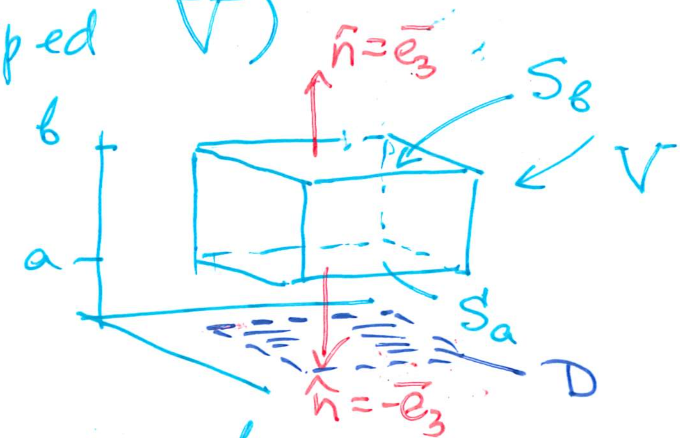
nablasymbolen

Gauss Sats. Om $V \subset \mathbb{R}^3$, $S = \partial V$ (Randen), $\bar{F}$ är C^1 i V med randen $\Rightarrow$

$$\underbrace{\iiint_V \operatorname{div} \bar{F} \underbrace{dxdydz}_{dV}}_{\text{V.L.}} = \underbrace{\iint_S \bar{F}\, d\bar{S}}_{\text{H.L.}}, \quad d\bar{S} \text{ orienterad}$$

med utåtriktad normal

Bevis (för en parallellepiped V)



$$\text{V.L.} = \iiint_V \left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} \right) dxdydz$$

$= I_1 + I_2 + I_3$, där

$$I_3 = \iiint_V \frac{\partial F_3}{\partial z} dxdydz = \left| \begin{array}{c} \text{stavar} \\ \text{i } z\text{-led} \end{array} \right| = \iint_D \left(\int_a^b \frac{\partial F_3}{\partial z} dz \right) dxdy =$$

$$= \iint_D \left(F_3(x,y,b) - F_3(x,y,a) \right) dxdy = \iint_{S_b} F_3(x,y,b)\, dS - \iint_{S_a} F_3(x,y,a)\, dS$$

$$= \iint_{S_b} \bar{F} \cdot \hat{n}\, dS + \iint_{S_a} \bar{F} \cdot \hat{n}\, dS$$

Analogt: I_1, I_2, I_3, vilket ger

$$\iiint_V \operatorname{div} \bar{F}\, dV = \oiint_{\partial V} \bar{F} \cdot d\bar{S} = \iint_S \bar{F} \cdot \hat{n}\, dS$$

Fysikaliska motiveringar / slutsatser:

Låt V vara en liten volym kring en punkt $\bar{a}$:

$\exists\, \bar{x}_0$ sådan att

$$\iiint_V \underbrace{\operatorname{div} \bar{F}(x)}_{\text{en kontinuerlig funktion}}\, dV = \operatorname{div} F(\bar{x}_0) \cdot \iiint_V dV$$

krymper till $\bar{a}$

$\bar{x}_0$ - ngn punkt

$$= \operatorname{div} F(\bar{x}_0) \cdot \operatorname{Vol}(V)$$

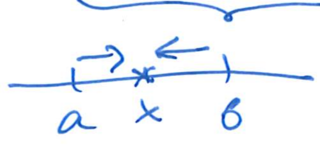
$$\Rightarrow \lim_{\operatorname{Vol}(V)\to 0} \frac{1}{\operatorname{Vol}(V)} \iiint_V \operatorname{div} F(x)\, dV = \operatorname{div} \bar{F}(\bar{x}) = \Big/ \begin{matrix}\text{Gauss}\\ \text{sats}\end{matrix} \Big/$$

$$= \lim_{\operatorname{Vol}(V)\to 0} \frac{1}{\operatorname{Vol}(V)} \iint_{\partial V} \bar{F} \cdot d\bar{S} = \text{H.L.}$$

Slutsats: Divergensen "visar/säger" något om hur mycket "vätska" med hastighetsflödet $\bar{F}$ flödar ut av (eller in i) punkten $\bar{x}$ per volymenhet.

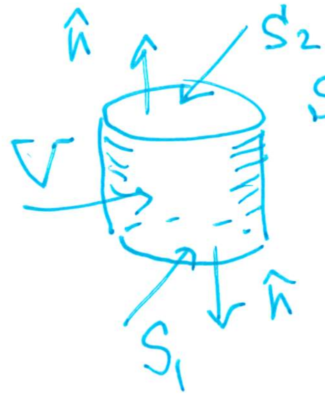
$\boxed{\text{Jmf:}}$ I en variabel:

$$\underbrace{\frac{1}{b-a}}_{|\Delta|} \underbrace{\int_a^b f(x)\, dx}_{F\ dS} \to f(x), \text{ om } \underbrace{b-a\to 0,\ a, b\to x}$$



Exempel (4.13)* $\quad \bar{A} = xyz(x,y,z)$,

S ges av $x^2+y^2=1$, $\quad 0 \le z \le 1$



S = mantel

Lösning: Observera att

$$V = \{\bar{r} : x^2+y^2 \le 1,\ 0 \le z \le 1\}$$

$$\iiint_V \operatorname{div}\bar{A} = \Big(\underbrace{\iint_S}_{\text{söks}} + \iint_{S_1} + \iint_{S_2}\Big)\bar{A}\cdot d\bar{S}$$

- $\operatorname{div}\bar{A} = \dfrac{\partial(x^2yz)}{\partial x} + \dfrac{\partial(xy^2z)}{\partial y} + \dfrac{\partial(xyz^2)}{\partial z} = 6xyz$

- $\displaystyle\iiint_V \operatorname{div}\bar{A} = \left|\begin{matrix}\text{skivor}\\ \text{i } z\text{-led}\end{matrix}\right| = \int_0^1\Big(\iint_{D_z} 6xyz\,dx\,dy\Big)dz =$

$$= \int_0^1 6z\Big(\int_0^{2\pi}\int_0^1 \rho^2 \underbrace{\cos\varphi\sin\varphi}_{\frac{1}{2}\sin 2\varphi}\cdot\rho\cdot d\rho\, d\varphi\Big)dz = 0$$

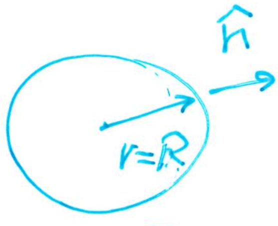
- $\displaystyle\iint_{S_1} \bar{A}\cdot\hat{n} = \iint_{S_1}\bar{A}\cdot(-\bar{e}_3)\,dS = -\iint_{S_1} xy\overset{z=0}{z^2}\,dS = 0$

- $\displaystyle\iint_{S_2} \bar{A}\cdot\hat{n} = \iint_{S_2}\bar{A}\cdot(\bar{e}_3)\,dS = \iint_{S_2} xy\overset{z=1}{z^2}\,dS = \iint_{S_2} xy\,dS = 0$

Alltså: $\displaystyle\iint_S \bar{A}\cdot\hat{n}\,dS = 0 - 0 - 0 = 0.$ ∎

Exempel 2. $\bar F = \dfrac{\bar r}{r^3} = \dfrac{1}{r^3}(x,y,z)$, $S: x^2+y^2+z^2=R^2$ (4)

Observera att:



$$\hat n = \frac{\bar r}{r} = \frac{1}{R}\bar r$$

$$\iint_S \bar F \cdot d\bar S = \iint_S \frac{1}{R^3}\bar r \cdot \frac{1}{R}\bar r \, dS =$$

$$= \iint_S \frac{R^2}{R^4} dS = \frac{1}{R^2} \text{Area}(S) = \frac{4\pi R^2}{R^2} = 4\pi$$

Å andra sidan : $\boxed{\operatorname{div} \varphi \bar F = \nabla\varphi \cdot \bar F + \varphi \operatorname{div} \bar F}$

$$\operatorname{div} \frac{\bar r}{r^3} = \nabla(r^{-3}) \cdot \bar r + r^{-3} \underbrace{\operatorname{div} \bar r}_{3} = -3r^{-4} \nabla r \cdot \bar r + \frac{3}{r^3} =$$

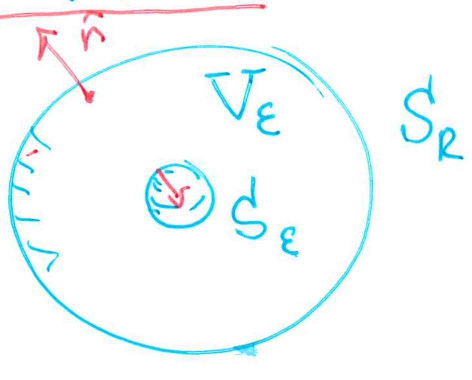
$$= -\frac{3}{r^4} \cdot \frac{\bar r \cdot \bar r}{r} + \frac{3}{r^3} = 0$$

vilket ger ($V_0 = \{x^2+y^2+z^2 \leq 1\}$ utan origo)

$$\iiint_{V_0} \operatorname{div} \frac{\bar r}{r^3} \, dxdydz = \iiint_{V_0} 0 \, dxdydz = 0 \, !$$

Varför stämmer inte? För att $\bar F \notin C^1(B)$

Gauss sats gäller för B med ett hål i singularitet :



$$\iiint_{V_\varepsilon} \operatorname{div} \bar F \, dV = \iint_{\partial V_\varepsilon} \bar F \cdot d\bar S =$$

$$= \iint_{S_R} \bar F \cdot d\bar S + \iint_{S_\varepsilon} \bar F \cdot d\bar S$$

d.v.s trippelintegral $= 0$ (p.g.a. $\operatorname{div}\bar{F} = 0$)

men

$$\iint_{S_\varepsilon} \bar{F}\cdot dS = \Big/ \hat{n} = -\frac{\bar{r}}{\varepsilon} \Big/ = \iint_{S_\varepsilon} \frac{\bar{r}}{\varepsilon^3}\cdot\left(-\frac{\bar{r}}{\varepsilon}\right) dS =$$

$$= -\iint_{S_\varepsilon} \frac{|\bar{r}|^2}{\varepsilon^4}\, dS = -\frac{\varepsilon^2}{\varepsilon^4}\iint_{S_\varepsilon} dS = -\frac{1}{\varepsilon^2}\,\mathrm{Area}(S_\varepsilon) = -\frac{4\pi\varepsilon^2}{\varepsilon^2}$$

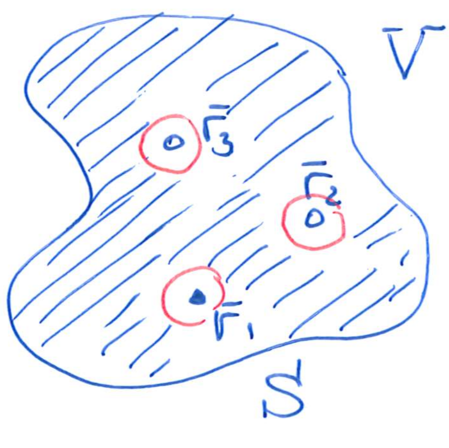
$$= -4\pi .$$

D.V.S.

$$\underbrace{\iiint \operatorname{div}\bar{F}\, dV}_{= 0} = \underbrace{\iint_{S_R} \bar{F}\cdot d\bar{S}}_{= 4\pi} + \underbrace{\iint_{S_\varepsilon} \bar{F}\cdot d\bar{S}}_{= -4\pi}$$

Metoden: Låt $V \subset \mathbb{R}^3$ vara en kropp (ett område), $\bar{F} \in C^1(V \setminus \{\bar{r}_1, \dots, \bar{r}_n\})$

a) Välj rimliga område V_k kring singulariteter $\bar{r}_1, \bar{r}_2, \dots \bar{r}_n$ så att $\bar{F}\cdot\hat{n}$ räknas enklast



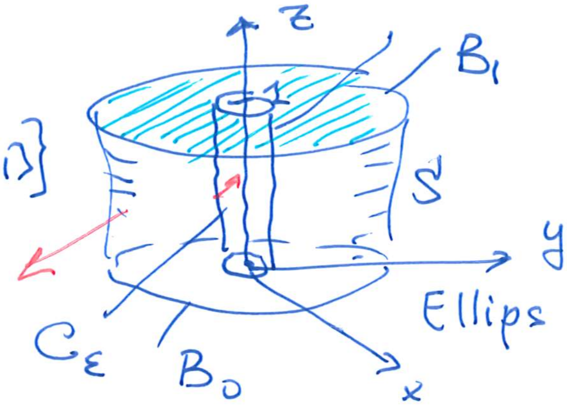
b) Använd Gauss sats i $V \setminus (V_1 \cup \dots \cup V_n)$

c) Om ytterliggre $\operatorname{div}\bar{F} = 0$ så gäller att

$$\iint_S \bar{F}\cdot\hat{n}\, dS = \iint_{S_1} \bar{F}\cdot\hat{n}\, dS + \dots + \iint_{S_n} \bar{F}\cdot\hat{n}\, dS$$

normaler utåt riktade.

Exempel 3* $\bar{F} = \frac{1}{x^2+y^2}(x, y, 0)$, S ges av $4x^2+9y^2=1$, $0 \le z \le 1$, $\hat{n}$ utåtriktad.



Lösning: Betrakta

$$V_\varepsilon: \{4x^2+9y^2 \le 1,\ x^2+y^2 \ge \varepsilon^2,\ z \in [0,1]\}$$

$$\partial V = S + B_0 + B_1 + C_\varepsilon$$

Observera att:

- $$\operatorname{div} \bar{F} = \frac{\partial}{\partial x}\left(\frac{x}{x^2+y^2}\right) + \frac{\partial}{\partial y}\left(\frac{y}{x^2+y^2}\right) + 0 = \frac{y^2-x^2}{(x^2+y^2)^2} + \frac{x^2-y^2}{(x^2+y^2)^2} = \underline{\underline{0}}$$

Alltså $$\iiint_{V_\varepsilon} \operatorname{div} \bar{F}\, dV = 0 \Rightarrow \iint_S \bar{F} \cdot d\bar{S} = -\left(\iint_{B_0} + \iint_{B_1} + \iint_{C_\varepsilon}\right) \bar{F}\, dS$$

- $$\iint_{B_0} \bar{F} \cdot \hat{n}\, dS = \iint_{B_0} \bar{F} \cdot (-\bar{e}_3)\, dS = 0 \qquad (\hat{n} = -\bar{e}_3)$$

- $$\iint_{B_1} \bar{F} \cdot \hat{n}\, dS = \iint_{B_1} \bar{F} \cdot (\bar{e}_3)\, dS = 0 \qquad (\hat{n} = e_3)$$

- $$\iint_{C_\varepsilon} \bar{F} \cdot \hat{n}\, dS = \left[\ C_\varepsilon \text{ ges av: } \bar{r} = \begin{pmatrix} \varepsilon \cos\varphi \\ \varepsilon \sin\varphi \\ z \end{pmatrix} \quad \underbrace{\begin{matrix} 0 \le \varphi \le 2\pi \\ 0 \le z \le 1 \end{matrix}}_{D}\ \right]$$

normalen:

$$\bar{n} = \bar{r}'_\varphi \times \bar{r}'_z = \varepsilon(\cos\varphi, \sin\varphi, 0)$$

$$\bar{n} = -\varepsilon(\cos\varphi, \sin\varphi, 0)$$

"inåtriktad"

$$= \iint_D \underbrace{\frac{1}{\varepsilon^2}\begin{pmatrix} \varepsilon\cos\varphi \\ \varepsilon\sin\varphi \\ 0 \end{pmatrix}}_{F} \cdot \underbrace{\begin{pmatrix} -\varepsilon\cos\varphi \\ -\varepsilon\sin\varphi \\ 0 \end{pmatrix} d\varphi\, dz}_{(\pm)\, \bar{r}'_\varphi \times \bar{r}'_z\, d\varphi\, dz} = -\frac{\varepsilon^2}{\varepsilon^2} \int_0^{2\pi}\!\!\int_0^1 1\, d\varphi\, dz = -2\pi$$

Svar: $$\iint_S \bar{F}\, d\bar{S} = 2\pi$$