Fö 7 Koordinattransformationer. Kroklinjiga koordinatsystem.

Betrakta $\mathbb{R}^3$, $\underline{e} = (\bar{e}_1\ \bar{e}_2\ \bar{e}_3)$ och $\underline{f} = (\bar{f}_1\ \bar{f}_2\ \bar{f}_3)$ två ON (= ortonormerade) baser. Då gäller

$$\underline{f} = \underline{e}\,T \quad (T = \text{transformationsmatris})$$

där $T^tT = TT^t = I$ (enhetsmatris), d.v.s. T är en <u>isometri</u> av $\mathbb{R}^3$: $T\bar{x} \cdot T\bar{x} = \bar{x} \cdot \bar{x}$.

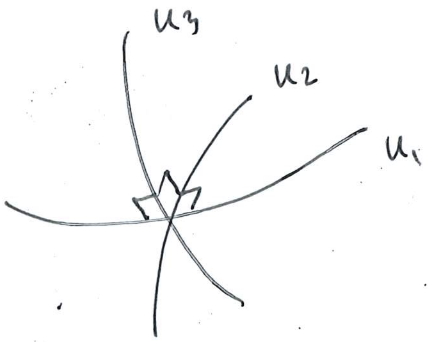
Vi kommer att generalisera det för en allmän C^1-avbildning, d.v.s

$$\begin{cases} u_1 = f_1(x,y,z) \\ u_2 = f_2(x,y,z) \\ u_3 = f_3(x,y,z) \end{cases} : \mathbb{R}^3 \to \mathbb{R}^3 \quad \underline{\text{en injektion}}$$

↑ ursprungligt <u>kartesiskt</u> <u>KS</u>

("<u>koordinatsystem</u>")

Def Ett kroklinjigt KS är <u>ortogonalt</u> om dess "koordinatlinjer" skär varandra under räta vinklar i varje punkt



"u_1" ~ "f_2 och f_3 = konst"

Cylindriskt KS:

$$F: \begin{cases} x = \rho\cos\varphi \\ y = \rho\sin\varphi \\ z = z \end{cases}$$

$(x, y, z) \longleftrightarrow (\rho, \varphi, z)$

där $\rho \in [0, \infty[$

$\varphi \in [0, 2\pi[$

$z \in \mathbb{R}$

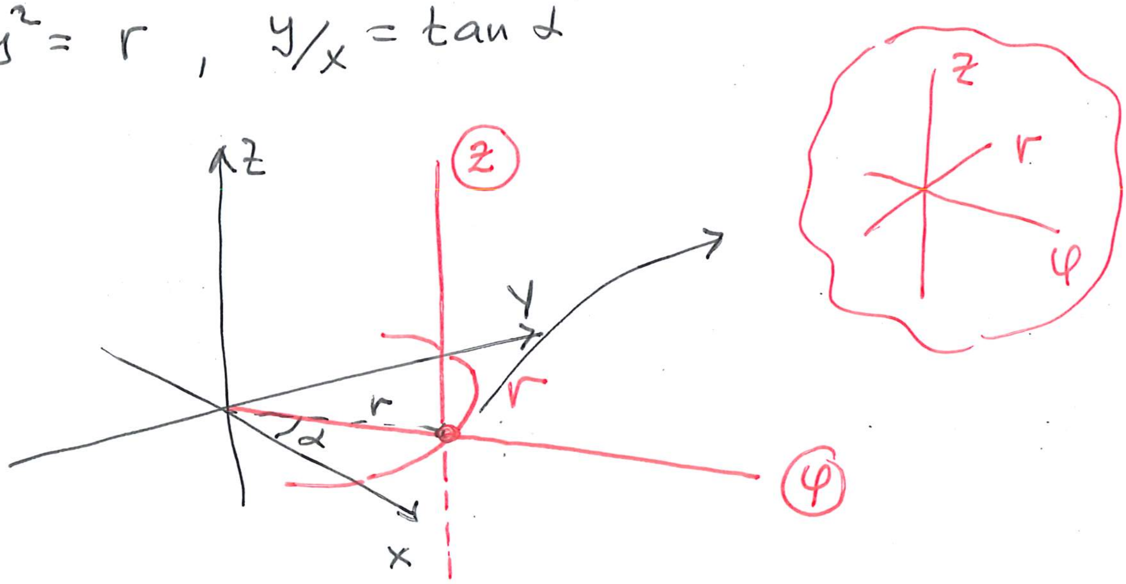
$\Downarrow$

$$\begin{cases} \rho = \sqrt{x^2+y^2} \\ \tan\varphi = y/x \\ z = z \end{cases}$$

(invers).

- z - koordinatlinje: $\rho = r_1$ och $\varphi = \alpha$:

$x^2+y^2 = r$, $y/x = \tan\alpha$



$$\begin{cases} \frac{d\bar{r}}{d\rho} = (\cos\varphi, \sin\varphi, 0) \\ \frac{d\bar{r}}{d\varphi} = (-\rho\sin\varphi, \rho\cos\varphi; 0) \\ \frac{d\bar{r}}{dz} = (0; 0; 1) \end{cases}$$

parvist ortogonala

Sats 1 Ett kroklinjig KS (u_1, u_2, u_3) är ortogonalt $\Longleftrightarrow$

$$\frac{\partial \bar{r}}{\partial u_i} \cdot \frac{\partial \bar{r}}{\partial u_j} = 0 \quad \text{för } i \neq j \text{ överallt.}$$

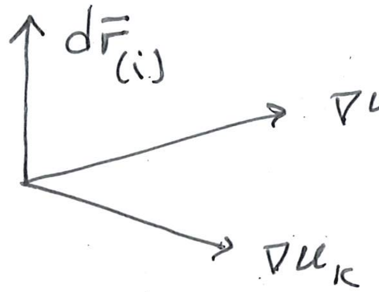
$$\Longleftrightarrow \quad \nabla u_i \cdot \nabla u_j = 0 \quad \forall\, i \neq j \quad \text{överallt.}$$

u_i-koordinatlinjen (K.L.) ges $u_k, u_j = \text{fixt}$.

d.v.s $du_k = du_j = 0 \Longleftrightarrow \nabla u_k \cdot d\bar{r}_{(i)} = \nabla u_j \cdot d\bar{r}_{(i)} = 0$

$\Rightarrow \quad d\bar{r}_{(i)} \parallel (\nabla u_k \times \nabla u_j)$ längs u_i-koordinatlinja.

För alla triplar (i, j, k)



$\Rightarrow \nabla u_1 \perp \nabla u_2 \perp u_3$

Också: $\dfrac{\partial \bar{r}}{\partial u_i} \parallel u_i$-(K.L.)

Def Man tillordnar ett ortogonalt, kroklinjigt K.S. ett ortonormerat basvectorsystem $\{\bar{e}_1, \bar{e}_2, \bar{e}_3\}$ enligt

$$\bar{e}_i = \frac{1}{h_i} \cdot \frac{\partial \bar{r}}{\partial u_i}, \quad h_i := \left| \frac{\partial \bar{r}}{\partial u_i} \right|$$

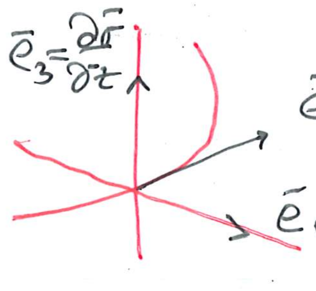
"normerad"

$(|\bar{e}_i| = 1$, d.v.s enhetsvectorn$)$

Obs. att $\bar{e}_i$ är en enhetstangentvektor till u_i-K.L.

Cylindriskt. KS

$h_1 = 1$
$h_2 = \rho$
$h_3 = 1$



$$\left.\begin{aligned} \bar{e}_2 &= \frac{d\bar{r}}{d\varphi} \\ \bar{e}_1 &= \frac{d\bar{r}}{d\rho} \end{aligned}\right\} \text{normerad!}$$

$\bar{e}_1 = (\cos\varphi, \sin\varphi, 0)$
$\bar{e}_2 = (-\sin\varphi, \cos\varphi, 0)$
$\bar{e}_3 = (0; 0; 1)$

Sats 2 Ortsvektordifferentialen

$$d\bar{r} = h_1 du_1\, \bar{e}_1 + h_2 du_2 \cdot \bar{e}_2 + h_3 du_3 \cdot \bar{e}_3$$

Bevis Kedjeregeln:

$$d\bar{r} = \underbrace{\frac{\partial \bar{r}}{\partial u_1}}_{h_1 \bar{e}_1} du_1 + \frac{\partial \bar{r}}{\partial u_2} du_2 + \frac{\partial \bar{r}}{\partial u_3} du_3$$

etc. ∎

Linjeelement i ett kroklinjigt ortogonalt K.S.

$$ds^2 = d\bar{r} \cdot d\bar{r} = h_1^2 du_1^2 + h_2^2 du_2^2 + h_3^2 du_3^2$$

Räkneregler.

Ett vektorfält

$$\bar{A} = \bar{A}(u_1, u_2, u_3) = \sum_{i=1}^{3} A_i(u_1, u_2, u_3)\, \bar{e}_i$$

$\boxed{\text{OBS}}$!

inte $\hat{x}, \hat{y}, \hat{z}$!

- $$\bar{A} \cdot \bar{B} = \sum_{i=1}^{3} A_i B_i$$
- $$\bar{A} \times \bar{B} = \begin{vmatrix} \bar{e}_1 & \bar{e}_2 & \bar{e}_3 \\ A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \end{vmatrix}$$

Add. Sfäriskt K.S.

$$\bar{r} = \begin{cases} x = r\cos\varphi\sin\theta \\ y = r\sin\varphi\sin\theta \\ z = r\cos\theta \end{cases} \qquad \begin{array}{l} r \in [0, \infty[\\ \varphi \in [0, 2\pi[\\ \theta \in [0, \pi] \end{array}$$

$$\bar{r}'_r = (\cos\varphi\sin\varphi, \sin\varphi\sin\theta, \cos\theta) \qquad h_1 = 1$$

$$\bar{r}'_\theta = (r\cos\varphi\cos\theta, r\sin\varphi\cos\theta, -r\sin\theta) \qquad h_2 = r$$

$$\bar{r}'_\varphi = (-r\sin\varphi\sin\theta, r\cos\varphi\sin\theta; 0) \qquad h_3 = r\sin\theta$$

- $h_1 h_2 h_3 = r^2\sin\theta = \left(\text{OBS. } \frac{\partial(x,y,z)}{\partial(r,\theta,\varphi)} = r^2\sin\theta\right)$
- $\bar{h} = (1, r, r\sin\theta)$

Gradient: via defenition:

$$df = \nabla f \cdot d\bar{r} = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz$$

I fall om (u_1, u_2, u_3) är ett ortogonalt K.S.:

$$d\bar{r} = h_1\hat{e}_1 du_1 + h_2\hat{e}_2 du_2 + h_3\hat{e}_3 du_3 \text{, alltså}$$

$$df = \frac{\partial f}{\partial u_1}du_1 + \frac{\partial f}{\partial u_2}du_2 + \frac{\partial f}{\partial u_3}du_3 = \left(\frac{1}{h_1}\frac{\partial f}{\partial u_1}\hat{e}_1 + \frac{1}{h_2}\frac{\partial f}{\partial u_2}\hat{e}_2 + \frac{1}{h_3}\frac{\partial f}{\partial u_3}\hat{e}_3\right)\cdot d\bar{r}$$

$$\Rightarrow \boxed{\operatorname{grad} f = \nabla f = \frac{1}{h_1}\frac{\partial f}{\partial u_1}\hat{e}_1 + \frac{1}{h_2}\frac{\partial f}{\partial u_2}\hat{e}_2 + \frac{1}{h_3}\frac{\partial f}{\partial u_3}\hat{e}_3}$$

Räknelegler i ett Ortogonalt K.S.

- $$\nabla\varphi = \frac{1}{h_1}\frac{\partial\varphi}{\partial u_1}\hat{e}_1 + \frac{1}{h_2}\frac{\partial\varphi}{\partial u_2}\hat{e}_2 + \frac{1}{h_3}\frac{\partial\varphi}{\partial u_3}\hat{e}_3$$
- $$\operatorname{div}\bar{F} = \frac{1}{h_1h_2h_3}\left[\frac{\partial(A_1h_2h_3)}{\partial u_1} + \frac{\partial(h_1A_2h_3)}{\partial u_2} + \frac{\partial(h_1h_2A_3)}{\partial u_3}\right]$$
- $$\operatorname{rot}\bar{F} = \frac{1}{h_1h_2h_3}\begin{vmatrix} h_1\hat{e}_1 & h_2\hat{e}_2 & h_3\hat{e}_3 \\ \partial_{u_1} & \partial_{u_2} & \partial_{u_3} \\ h_1F_1 & h_2F_2 & h_3F_3 \end{vmatrix}$$

Exempel Bestäm $\operatorname{div}\bar{F}$ och $\operatorname{rot}\bar{F}$, $\bar{F} = \left(-\frac{y}{x^2+y^2}; \frac{x}{x^2+y^2}; 0\right)$

Lös. I Cylindriset K.S. : $\bar{h} = (1, \rho, 1)$,

$$\bar{F} = -\frac{y}{x^2+y^2}\hat{x} + \frac{x}{x^2+y^2}\hat{y} \;\textcircled{=}$$

en ON-matris (T)

$$(\hat{e}_\rho \;\; \hat{e}_\varphi \;\; \hat{e}_z) = (\hat{x}, \hat{y} \;\; \hat{z})\begin{pmatrix} \cos\varphi & -\sin\varphi & 0 \\ \sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix} \Rightarrow T^{-1} = T^t$$

$$(\hat{x} \;\; \hat{y} \;\; \hat{z}) = (\hat{e}_\rho \;\; \hat{e}_\varphi \;\; \hat{e}_z)\,T^t = (\cos\varphi\,\hat{e}_\rho - \sin\varphi\,\hat{e}_\varphi) + (\sin\varphi\,\hat{e}_\rho + \cos\varphi\,\hat{e}_\varphi) + \hat{z}$$

$$\Rightarrow \bar{F} = -\frac{\sin\varphi}{\rho}(\cos\varphi\,\hat{e}_\rho - \sin\varphi\,\hat{e}_\varphi) + \frac{\cos\varphi}{\rho}(\sin\varphi\,\hat{e}_\rho + \cos\varphi\,\hat{e}_\varphi) = \frac{1}{\rho}\hat{e}_\varphi$$

$$\operatorname{div}\bar{F} = \frac{1}{\rho}\left[\frac{\partial}{\partial\rho}0 + \frac{\partial}{\partial\varphi}\left(\frac{1}{\rho}\cdot 1\right) + \frac{\partial}{\partial z}(0)\right] = 0$$

$$\operatorname{rot}\bar{F} = \frac{1}{\rho}\begin{vmatrix} \hat{e}_\rho & \rho\hat{e}_\varphi & \hat{e}_z \\ \partial_\rho & \partial_\varphi & \partial_z \\ 0 & \underbrace{\rho\cdot\frac{1}{\rho}}_{=1} & 0 \end{vmatrix} = \bar{0}$$