Fö 8 Integraler i kroklinjiga koordinater

(A) Kurvintegraler $\int_\Gamma \bar{F}\cdot d\bar{r}$

(B) Ytintegraler $\iint_S \bar{F}\cdot\hat{n}\,dS = \iint_S \bar{F}\cdot d\bar{S}$

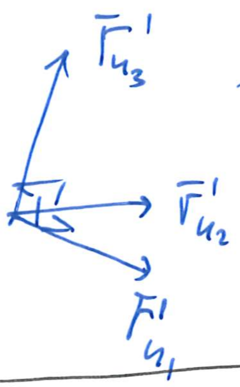
h_i Lamé koeff

(C) Volymintegraler $\iiint_V \Phi\,dxdydz$

(C) **Antag att:** $\bar{r} = (x(\bar{u}), y(\bar{u}), z(\bar{u}))$

$$\frac{d(x,y,z)}{d(u_1,u_2,u_3)} = \begin{vmatrix} x'_{u_1} & x'_{u_2} & x'_{u_3} \\ y'_{u_1} & y'_{u_2} & y'_{u_3} \\ \underbrace{z'_{u_1}}_{\bar{r}'_{u_1}} & \underbrace{z'_{u_2}}_{\bar{r}'_{u_2}} & \underbrace{z'_{u_3}}_{\bar{r}'_{u_3}} \end{vmatrix} = \bar{r}'_{u_1}\cdot(\bar{r}'_{u_2}\times\bar{r}'_{u_3}) =$$

= volym av parallellepiped med sidorna $\bar{r}'_{u_i}$:



← alla vinklar = 90°

$|\bar{r}'_{u_i}| = h_i$

$\Rightarrow \boxed{Vol = h_1 h_2 h_3}$

$$\boxed{\iiint_V \underbrace{\Phi(x,y,z)}_{F}\,dxdydz = \iiint_E \Phi(\bar{r}(u))\,h_1h_2h_3\,du_1du_2du_3}$$

Ex. Sfäriska koordinater: $h_1h_2h_3 = 1\cdot r\cdot r\sin\theta = r^2\sin\theta$

(som tidigare i TATA43)

(A) Kurvintegraler:

$\Gamma: \quad \bar{u} : [a,b] \to \mathbb{R}^3$

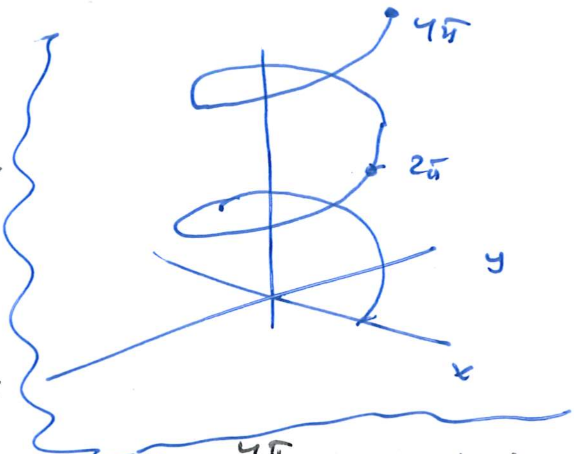
$$\int_\Gamma \bar{F}\cdot d\bar{r} = \int_a^b \bar{F}(\bar{u}(t))\cdot \bar{u}'(t)\,dt \ \circleddash$$

- $d\bar{r} = \sum \bar{r}'_{u_i}\, du_i = \sum h_i \hat{e}_i\, du_i$
- $\bar{F} = \sum F_i(u)\, \hat{e}_i$

$$\circleddash \ \boxed{\int_a^b \left(\sum_{i=1}^{3} F_i(\bar{u}(t))\, h_i \frac{du_i}{dt}\right) dt.}$$

Ex. Beräkna $\int_\Gamma \bar{F}\cdot d\bar{r}$, där Γ i cylindriska koordinater ges av: $\bar{u}:\{\rho = 2, \varphi = t, z = t\}$ $t = [0, 4\pi]$

och $\bar{F}(\rho,\varphi,z) = \frac{2z}{z^2+4}\hat{z} + \varphi\hat{\varphi}$.



Lös. $h_1 = h_3 = 1, \quad h_2 = \rho$.

$$d\bar{r} = \hat{\rho}\,d\rho + \rho\hat{\varphi}\,d\varphi + \hat{z}\,dz =$$
$$= 0 + 2\hat{\varphi}\,dt + \hat{z}\,dt = (2\hat{\varphi} + \hat{z})\,dt$$

$$\int_\Gamma \bar{F}\cdot d\bar{r} = \int_0^{4\pi} \left(\varphi\hat{\varphi} + \frac{2z}{z^2+4}\hat{z}\right)\cdot(2\hat{\varphi}+\hat{z})\,dt = \int_0^{4\pi}\left(2t + \frac{2t}{t^2+4}\right)dt$$

$$I = \int_0^{4\pi}\left(2t + \frac{2t}{t^2+4}\right)dt = \left[t^2 + \ln(t^2+4)\right]_0^{4\pi} = (4\pi)^2 + \ln\frac{16\pi^2+4}{4}$$

$$= 16\pi^2 + \ln(4\pi^2+1)$$

Alternativ II

$\bar{F} = \varphi \hat{e}_\varphi + \frac{2z}{z^2+4}\hat{e}_z =$ / bak to kartesianska KS /

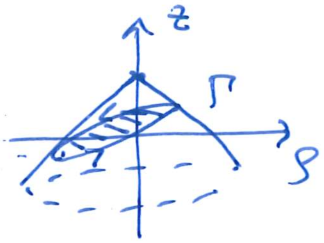
$= \varphi\cdot(-\sin\varphi, \cos\varphi, 0) + \frac{2z}{z^2+4}(0,0,1) =$

$= \left(-\varphi\sin\varphi, \varphi\cos\varphi, \frac{2z}{z^2+4}\right) = (\Gamma) = \left(-t\sin t, t\cos t, \frac{2t}{t^2+4}\right)$

$$I = \int_0^{4\pi} \left(-t\sin t, t\cos t, \frac{2t}{t^2+4}\right)\cdot \frac{d}{dt}(2\cos t, 2\sin t, t)\,dt =$$

$$= \dots = \int_0^{4\pi}\left(2t + \frac{2t}{t^2+4}\right)dt$$

8.21) Beräkna $\int_\Gamma \bar{A}\cdot d\bar{r}$, $\bar{A}(\rho,\varphi,z) = \rho\hat{\varphi}$ och Γ = skärningskurvan mellan $y+2z=1$ och konen $\rho+z=2$ (orientering moturs från $(0,0,100)$)



Lös $\begin{cases} z = 2-\rho \quad : \\ y+2z=1 \text{ - ett plan}\end{cases}$

$y = \rho\sin\varphi$: Γ ges av $\begin{cases} z=2-\rho\\ 2z+\rho\sin\varphi = 1\end{cases}$

$\Rightarrow 4-2\rho+\rho\sin\varphi = 1, \quad \rho = \frac{3}{2-\sin\varphi}$

$z = 2-\rho = 2-\frac{3}{2-\sin\varphi} = \frac{1-2\sin\varphi}{2-\sin\varphi}.$

$\bar{u}: \begin{cases} u_1=\rho \\ u_2=\varphi \\ u_3=z\end{cases}$

$\bar{A}\cdot d\bar{r} = \rho\hat{\varphi}\cdot(\hat{\rho}\, d\rho + \rho\hat{\varphi}\, d\varphi + \hat{z}\, dz) = \rho^2 d\varphi = \left(\frac{3}{2-\sin\varphi}\right)^2 d\varphi$

$$I = \int_0^{2\pi}\rho^2 d\varphi = \int_0^{2\pi}\frac{9}{(2-\sin\varphi)^2}d\varphi = \text{svårt } \left(t=\tan\frac{\varphi}{2} \dots\right).$$

Alternativ lösning: m.h.a. Stokes:

$$\oint_\Gamma \bar{A}\cdot d\bar{r} = \iint_S (\nabla\times\bar{A})\cdot d\bar{S} =$$

$$\operatorname{rot}\bar{A} = \text{i cylinderkoord.} = \frac{1}{1\cdot 1\cdot \rho}\begin{vmatrix} \hat{\rho} & \rho\hat{\varphi} & \hat{z} \\ \partial_\rho & \partial_\varphi & \partial_z \\ 0 & \rho\cdot\rho & 0 \end{vmatrix} = \frac{2\rho\hat{z}}{\rho} = 2\hat{z}$$

h_2 h_2 A_2

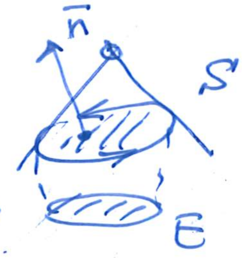
Altså $\nabla\times\bar{A} = 2\hat{z}$

S = del av planet $y+2z=1$ som fås från konen: $\rho = 2-z$, $\rho^2 = (2-z)^2 \iff$

$\iff x^2+y^2 = \left(\frac{3+y}{2}\right)^2 \iff 4x^2+3y^2-6y = 9 \iff$

$\iff \frac{x^2}{3} + \frac{(y-1)^2}{4} = 1$, d.v.s. $a=\sqrt{3}$, $b=\sqrt{4}=2$,

ellipsens AREA $:= \pi\cdot a\cdot b = \pi\cdot 2\sqrt{3}$.



Parameteriserar S:

$z = \frac{1-y}{2}$ $\qquad \bar{r} = \left(x, y, \frac{1-y}{2}\right)$

$\bar{n} = \bar{r}'_x \times \bar{r}'_y = (0, \tfrac{1}{2}; 1)$ (riktad uppåt ↑)

$$\Rightarrow \iint_S \operatorname{rot}\bar{A}\cdot d\bar{S} = \iint_S 2\hat{z}\cdot d\bar{S} = \iint_E 2(0,0,1)\cdot(0,\tfrac{1}{2};1)\,dxdy$$

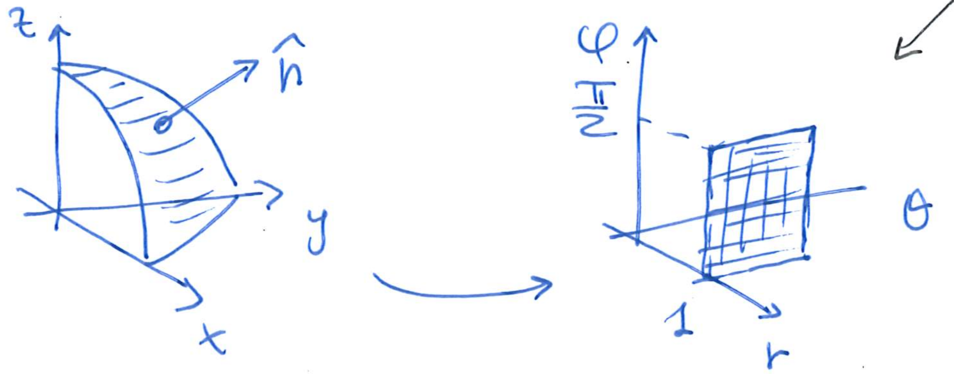
$$= 2\iint_E dxdy = 2\cdot(2\sqrt{3}\pi) = 4\sqrt{3}\pi$$

B Ytintegraler.

Antar att S är en del av $u_3 = \text{konst} = a$ där $(u_1, u_2) \in E$. (koordinatyta)

Ex. B1 Sfäriska koordinater: $x, y, z \ge 0$;

$x^2+y^2+z^2 = 1$ ger: $\underbrace{r = 1,\ 0 \le \theta \le \frac{\pi}{2},\ 0 \le \varphi \le \frac{\pi}{2}}$



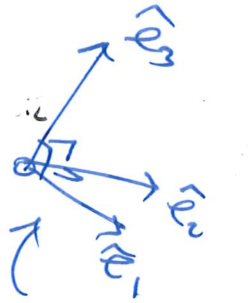
Söker flödet av $\bar{A} = A_1\hat{e}_1 + A_2\hat{e}_2 + A_3\hat{e}_3$:

- $$\iint_S \bar{A}\cdot\hat{n}\,dS = \iint_E \bar{A}\cdot(\bar{r}_{u_1}\times\bar{r}_{u_2})\,du_1du_2 =$$

$d\bar{S} = h_1h_2\bar{e}_3\,du_1du_2$

$$= \iint_E \bar{A}\cdot(h_1\hat{e}_1\times h_2\hat{e}_2)\,du_1du_2 =$$

$$= \iint_E \underbrace{(\bar{A}\cdot\bar{e}_3)}_{=A_3} h_1h_2\,du_1du_2$$



H.O. ON-bas

Ex B1 forts. $\bar{A} = r\cdot\hat{r}$, S utåtriktad, S ges av $\underbrace{r = u_1 = 1}$

$$\iint_S \bar{A}\cdot d\bar{S} = \iint_E r\cdot\hat{r}\cdot\underbrace{\underset{\hat{r}}{\hat{e}_1}\ \underset{r}{h_2}\ \underset{r\sin\theta}{h_3}\,du_2du_3} = \int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}} \underbrace{r\cdot r^2}_{=1}\sin\theta\,d\varphi\,d\theta = \frac{\pi}{2}$$

Basvectornas differentialer

T.ex. i cylindriska koordinater:

$$\hat{\rho} = (\cos\varphi, \sin\varphi, 0)$$
$$\hat{\varphi} = (-\sin\varphi, \cos\varphi, 0)$$
$$\hat{z} = (0, 0, 1)$$

- $d\hat{\rho} = \underbrace{\frac{\partial\hat{\rho}}{\partial\rho}}_{=0} d\rho + \frac{\partial\hat{\rho}}{\partial\varphi} d\varphi + \underbrace{\frac{\partial\hat{\rho}}{\partial z}}_{=0} dz = (-\sin\varphi, \cos\varphi, 0)\,d\varphi = \hat{\varphi}\,d\varphi$
- $d\hat{\varphi} = (-\cos\varphi, -\sin\varphi, 0)\,d\varphi = -\hat{\rho}\,d\varphi$
- $d\hat{z} = 0$

<u>8.10</u> Bestäm tangentvectorn i (x,y,z) till

$\Gamma:\ r = 2\sin 2\theta,\ \varphi = \frac{\pi}{4},\ x = y = \frac{\sqrt{6}}{4}, z = \frac{3}{2}.$ $\left(\Leftrightarrow r = \sqrt{\frac{6}{16}\cdot 2 + \frac{9}{4}}\right.$

$r = \sqrt{3}$

$\theta = \arccos\frac{\sqrt{3}}{2} = \frac{\pi}{6}.$

<u>Lösning</u> / Sfäriska koordinater

$$\begin{cases} d\hat{r} = \hat{\theta}\,d\theta + \sin\theta\,\hat{\varphi}\,d\varphi \\ d\hat{\theta} = -\hat{r}\,d\theta + \cos\theta\,\hat{\varphi}\,d\varphi \\ d\hat{\varphi} = -(\sin\theta\,\hat{r} + \cos\theta\cdot\hat{\theta})\,d\varphi \end{cases}$$

$\bar{r} = r\hat{r}$

$$\bar{r} = r\hat{r} = 2\sin 2\theta\,\hat{r}$$

$$\frac{d\bar{r}}{d\theta} = 4\cos 2\theta\cdot\hat{r} + 2\sin 2\theta\,\frac{d\hat{r}}{d\theta} = 4\cos 2\theta\,\hat{r} + 2\sin 2\theta\cdot\hat{\theta} =$$
$$= 2\hat{r} + \sqrt{3}\,\hat{\theta}$$

$$\hat{r} = \hat{x}\cos\varphi\sin\theta + \sin\varphi\sin\theta\,\hat{y} + \cos\theta\,\hat{z} = \frac{\sqrt{2}}{4}(\hat{x}+\hat{y}) + \frac{\sqrt{3}}{2}\hat{z}$$

$$\hat{\theta} = \hat{x}\cos\varphi\cos\theta + \hat{y}\sin\varphi\cos\theta + -\sin\theta\,\hat{z} = \frac{\sqrt{3}}{2}\frac{\sqrt{2}}{2}(\hat{x}+\hat{y}) - \frac{1}{2}\hat{z}$$

$$\frac{d\bar{r}}{d\theta} = \left(\frac{\sqrt{2}}{2} + \frac{3}{2}\frac{\sqrt{2}}{2}\right)(\hat{x}+\hat{y}) + \left(\sqrt{3} - \frac{\sqrt{3}}{2}\right)\hat{z} = \frac{5\sqrt{2}}{2}(\hat{x}+\hat{y}) + \frac{\sqrt{3}}{2}\hat{z}$$