Fö 9 Repetition

Allmäna frågor.

Kurvintegraler:

- $\displaystyle\int_\gamma \bar{F}\cdot d\bar{r} = \int_a^b F(\bar{u})\cdot \frac{d\bar{r}}{dt}\cdot dt$

- Orientering

- $\displaystyle\iint_S \bar{F}\cdot d\bar{S} \overset{\text{def}}{=} \iint_S \bar{F}\cdot\hat{n}\, dS = \iint_E \bar{F}\cdot(\bar{r}'_{u_1}\times\bar{r}'_{u_2})\, du_1 du_2$

- Kroklinjig K.S.:

$$\frac{\partial\bar{r}}{\partial u_i} = \hat{e}_i\cdot\left|\frac{\partial\bar{r}}{\partial u_i}\right| = \hat{e}_i h_i$$

Alternativt:

$(u_1, u_2, u_3) = (u_i(x,y,z), \ldots \text{etc})$, så gäller

$\{u_i\}$ är ortogonalt $\Leftrightarrow$ $\nabla u_1 \perp \nabla u_2 \perp \nabla u_3$.

I vilket fall:

$|\nabla u_i| = 1/h_i$, eller $h_i = 1/|\nabla u_i|$

Exempel 1 | Sfäriskt koordinat system.

$$\begin{cases} x = r\cos\varphi\sin\theta \\ y = r\sin\varphi\sin\theta \\ z = r\cos\theta \end{cases} \sim \begin{cases} r = \sqrt{x^2+y^2+z^2} = u_1 \\ \theta = \arccos\frac{z}{r} = u_2 \\ \varphi = \arctan\frac{y}{x} = u_3 \end{cases}$$

Vilket går till exempell att

$$|\nabla u_1| = \left(\frac{x}{\sqrt{x^2+y^2+z^2}}, \frac{y}{\sqrt{x^2+y^2+z^2}}, \frac{z}{\sqrt{x^2+y^2+z^2}} \right), \quad h_1 = 1/1 = 1$$

$$\nabla u_2 = \nabla\theta = \frac{-1}{\sqrt{1-\frac{z^2}{r^2}}} \left(\frac{r\nabla z - z\nabla r}{r^2} \right) = \frac{-r\nabla z + z\nabla r}{r\sqrt{r^2-z^2}}$$

där $\nabla r = \frac{\vec{r}}{|\vec{r}|} = \hat{r}$ och $\nabla z = (0,0,1) = \hat{z}$, alltså

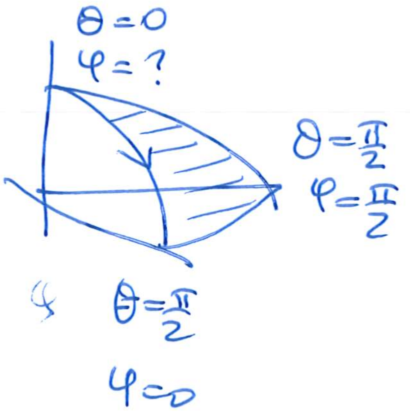
$$|\nabla\theta| = \frac{1}{r^2(r^2-z^2)} \cdot \left(r^2 \cdot 1 - 2zr \underbrace{\hat{r}\cdot\hat{z}}_{\frac{z}{r}} + z^2 \cdot 1 \right) = \frac{r^2-z^2}{r^2(r^2-z^2)}$$

$$\Rightarrow \quad h_2 = 1/|\nabla\theta| = 1/r \quad \text{etc}$$

Exempel 2. Räkna z-moment över S, d.v.s.

$$I = \iint_S z\,dS, \quad S \text{ ges av } x,y,z \geq 0, \quad x^2+y^2+z^2=1$$

Lösning. I sfäriskt k.s.: S ges av $0 \leq \varphi \leq \frac{\pi}{2}$, $0 \leq \theta \leq \frac{\pi}{2}$

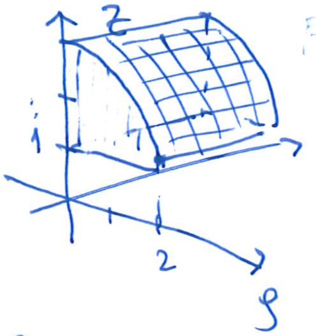


$$I = \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \underbrace{r\cos\theta}_{z} \cdot \underbrace{r}_{h_1} \cdot \underbrace{r\sin\theta}_{h_2} \, d\varphi\, d\theta = |r=1|$$

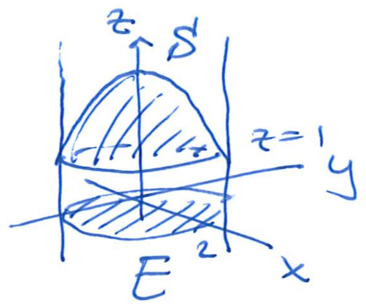
$$= \int_0^{\frac{\pi}{2}} d\varphi \cdot \int_0^{\frac{\pi}{2}} \cos\theta \sin\theta \, d\theta = \frac{\pi}{2} \cdot \left[\frac{\sin^2\theta}{2} \right]_0^{\frac{\pi}{2}} = \frac{\pi}{4}.$$

Exempel 3 Beräkna arean av den del av paraboloiden $z=5-x^2-y^2$ som innanför cylindeer $x^2+y^2=4$. (Tenta, 2015/08/26)

Lösning I cylindriskt koordinat system



ingen nivåytan



(I)

$$\begin{cases} z = 5-\rho^2 \\ \rho = 2 \end{cases} \Rightarrow z=1,\ \rho=2$$

$$\text{Area} = \iint_E \sqrt{1+|\nabla z|^2}\, dx\,dy = \iint_E \sqrt{1+4(x^2+y^2)}\, dx\,dy =$$

$$= \int_0^{2\pi}\int_0^{2} \sqrt{1+4\rho^2}\,\rho\, d\rho\, d\varphi = 2\pi\left[\frac{(1+4\rho^2)^{3/2}}{12}\right]_0^2 = \frac{\pi}{6}(17\sqrt{17}-1)$$

(II) I Cylindriskt K.S. $\bar{r} = \rho\hat{\rho} + z\hat{z} = \rho\hat{\rho} + (5-\rho^2)\hat{z}$,

där $(\rho,\varphi) \in [0,2]\times[0,2\pi]$.

$$\begin{cases} d\hat{\rho} = \hat{\varphi}\, d\varphi \\ d\hat{\varphi} = -\hat{\rho}\, d\varphi \\ d\hat{z} = 0 \end{cases} \Rightarrow d\bar{r} = \rho\hat{\varphi}\, d\varphi + \hat{\rho}\, d\rho + (-2\rho)\hat{z}\, d\rho$$

$$\frac{\partial \bar{r}}{\partial \rho}\times\frac{\partial \bar{r}}{\partial \varphi} = (\hat{\rho} - 2\rho\hat{z})\times(\rho\hat{\varphi}) = \rho\hat{z} + 2\rho^2\hat{\rho}$$

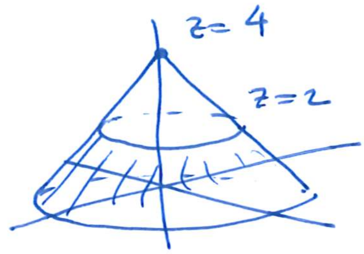
$$\left|\frac{\partial \bar{r}}{\partial \rho}\times\frac{\partial \bar{r}}{\partial \varphi}\right| = \sqrt{\rho^2+4\rho^4}$$

$$\text{Area} = \int_0^{2\pi}\int_0^{2} \sqrt{\rho^2+4\rho^4}\, d\rho\, d\varphi = \frac{\pi}{6}(17\sqrt{17}-1)$$

(Tenta 2015/08/26) Beräkna flödet av

$\bar{A}(x,y,z) = \frac{1}{(x^2+y^2+z^2)^{3/2}}(x,y,z)$ genom $z = 4-\sqrt{x^2+y^2}$

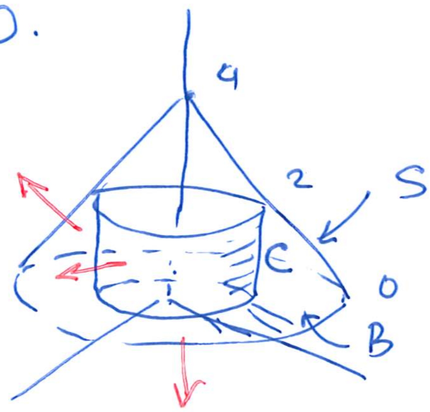
$0 \le z \le 2$.



Lösning $\bar{A} = \frac{1}{r^3}\cdot\bar{r} = \frac{r\hat{r}}{r^3} = \frac{\hat{r}}{r^2}$

i sfäriska koordinater $\Rightarrow$

$\operatorname{div} A = \frac{1}{r^2\sin\theta}\frac{\partial}{\partial r}\left(\frac{1}{r^2}\cdot r^2\sin\theta\right) = 0.$



Kroppen mellan S, C, B är K.

C: $\begin{cases} x^2+y^2=4 \\ 0\le z\le 2, \end{cases}$ (cylindriskt)

B: $\begin{cases} 4\le x^2+y^2\le 16 \\ z=0 \end{cases}$

Enligt Gauss sats (obs. orientering av C, utåt):

$$\iint\limits_{S-C+B} \bar{A}\,d\bar{S} = \iiint\limits_{K} \operatorname{div}\bar{A} = 0 \Rightarrow \iint\limits_{S}\bar{A}\,d\bar{S} = \underbrace{\iint\limits_{C}\bar{A}\,d\bar{S}}_{=I_1} - \underbrace{\iint\limits_{B}\bar{A}\,dS}_{=I_2}.$$

$I_1 = \left| \begin{array}{l} C \text{ är i cylindriskt KS:} \\ \rho = 2,\ z\in[0,2],\ \varphi\in[0,2\pi] \\ \bar{r} = \rho\hat{\rho} + z\hat{z} = 2\hat{\rho}+z\hat{z} \\ \bar{A} = \frac{1}{(4+z^2)^{3/2}}(2\hat{\rho}+z\hat{z}) \end{array} \right| = \int_0^{2\pi}\int_0^2 A_1 h_2 h_3\, du_2\, du_3 =$

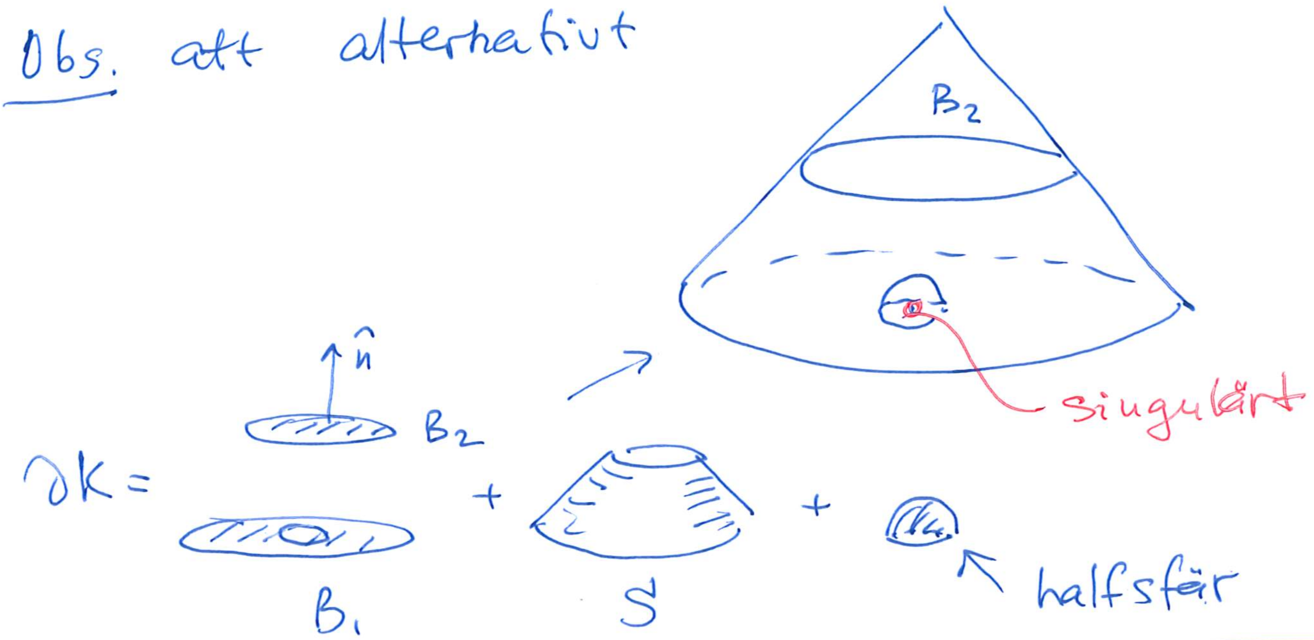
$= \int_0^{2\pi}\int_0^2 \frac{2\cdot 2\cdot dz\,d\varphi}{(4+z^2)^{3/2}} = 8\pi\int_0^2\frac{dz}{(z^2+4)^{3/2}}$

$= \left| \begin{array}{l} z = 2\tan t \\ 0\le t\le \frac{\pi}{4} \\ dz = \frac{2dt}{\cos^2 t} \end{array} \right| = 8\pi\int_0^{\frac{\pi}{4}} \frac{2dt}{\cos^2 t\cdot 8\frac{1}{\cos^3 t}} = 2\pi\int_0^{\frac{\pi}{4}}\cos t\,dt = \pi\sqrt{2}$

$$I_2 = \left| \begin{array}{l} B \text{ ges av } \quad z=0 \\ \bar{r} = \rho\hat{\rho}. \\ 0 \le \varphi \le 2\pi \quad 2 \le \rho \le 4 \end{array} \right| = \int_0^{2\pi}\int_2^4 A_3 h_1 h_2 \, du_1 du_2 = 0$$

<u>Svar</u> $I = \pi\sqrt{2}$.

<u>Obs.</u> att alternativt



Tenta 2017/08/23 | Bestäm alla funktioner $f(y,z)$ så att $A(x,y,z) = (2xy + f(y,z))\hat{x} + (x^2 + 2yz + 2x + z)\hat{y} + (y^2 + 2xz + y)\hat{z}$ blir ett potentialfält. Bestäm potentialen

<u>Lösn.</u> $(\text{rot}\,\bar{A})_x = \frac{\partial}{\partial y}(y^2 + 2xz + y) - \frac{\partial}{\partial z}(x^2 + 2yz + 2x + z) = 2y + 1 - 2y - 1 = 0$

$(\text{rot}\,\bar{A})_y = \frac{\partial}{\partial z}(2xy + f(y,z)) - \frac{\partial}{\partial x}(y^2 + 2xz + y) = \frac{\partial f(y,z)}{\partial z} - 2z = 0$

$(\text{rot}\,\bar{A})_z = \frac{\partial}{\partial x}(x^2 + 2yz + 2x + z) - \frac{\partial}{\partial y}(2xy + f(y,z)) = 2x + 2 - 2x - \frac{\partial f(y,z)}{\partial y} = 0$

$\Leftrightarrow \quad \frac{\partial f(y,z)}{\partial z} = 2z$ och $\frac{\partial f(y,z)}{\partial y} = 2 \quad \Rightarrow \quad f(y,z) = z^2 + h(y)$,

$\frac{\partial f(y,z)}{\partial y} = h'(y) = 2 \Rightarrow \quad h = 2y + C \quad \Rightarrow \quad \underline{f(y,z) = z^2 + 2y + C}$

Ny

$$\bar{A} = (2xy + z^2 + 2y + C)\hat{x} + (x^2 + 2yz + 2x + z)\hat{y} + (y^2 + 2xz + y)\hat{z}$$

$\bar{A} = \nabla \Phi$ ger

$$\Phi'_x = 2xy + z^2 + 2y + C \Rightarrow \Phi = x^2y + xz^2 + 2yx + Cx + H(y,z)$$

$$\Phi'_y = \underline{x^2 + 2x} + H'_y(y,z) = \underline{x^2} + 2yz + \underline{2x} + z \Rightarrow$$

$$\Rightarrow H'_y(y,z) = 2yz + z \Rightarrow H = y^2z + yz + h(z)$$

$$\Phi = x^2y + xz^2 + 2yx + Cx + y^2z + yz + h(z)$$

$$\Phi'_z = 2xz + y^2 + y + h' = 2xz + y^2 + y \Rightarrow h' = 0.$$

Svar $\Phi = x^2y + xz^2 + 2yx + y^2z + yz + C_1x + C_2$