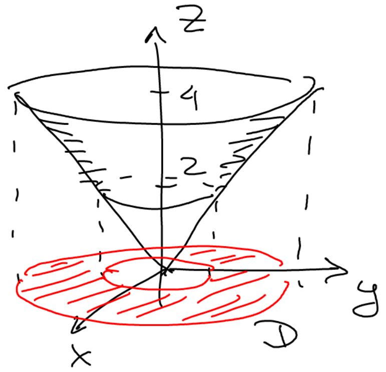
Tentamen TATA44, 2023-10-24
Lösningsförslag

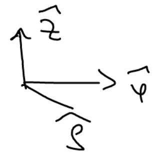



(1) **Alternativ 1:** i cylindriska koordinater

$\bar{r} = \rho\hat{\rho} + z\hat{z}$ där $\rho = \sqrt{x^2+y^2} = \frac{z}{2}$

$\bar{r} = \rho\hat{\rho} + 2\rho\hat{z}$, $\begin{cases} 2 \le z \le 4, & 1 \le \rho \le 2 \\ 0 \le \varphi \le 2\pi \end{cases}$

$d\bar{r} = \hat{\rho}\,d\rho + \rho\hat{\varphi}\,d\varphi + 2\hat{z}\,d\rho$, alltså $\frac{\partial \bar{r}}{\partial \rho} = \hat{\rho} + 2\hat{z}$, $\frac{\partial \bar{r}}{\partial \varphi} = \rho\hat{\varphi}$




$\Rightarrow \frac{\partial \bar{r}}{\partial z} \times \frac{\partial \bar{r}}{\partial \varphi} = (\hat{\rho} + 2\hat{z}) \times \rho\hat{\varphi} = \rho\hat{z} - 2\rho\hat{\rho}$

$\left|\frac{\partial \bar{r}}{\partial z} \times \frac{\partial \bar{r}}{\partial \varphi}\right| = \left(\rho^2 + (2\rho)^2\right)^{1/2} = \rho\sqrt{5} \quad \Rightarrow$

$$\iint_S x^2 z\,dS = \iint_D (\rho^2\cos^2\varphi)\cdot 2\rho\cdot\rho\sqrt{5}\,d\rho\,d\varphi = 2\sqrt{5}\int_0^{2\pi}\int_1^2 \rho^4\cos^2\varphi\,\rho\,d\rho\,d\varphi =$$

$$= 2\sqrt{5}\cdot\pi\cdot\left[\frac{\rho^5}{5}\right]_1^2 = \frac{62\sqrt{5}}{5}\pi$$

**Alternativ 2:** areaelement: $dS = \sqrt{1 + z_x'^2 + z_y'^2}\cdot dx\,dy$,

där $z = 2\sqrt{x^2+y^2} \Rightarrow dS = \sqrt{1 + 4\frac{x^2}{x^2+y^2} + 4\frac{y^2}{x^2+y^2}}\,dx\,dy = \sqrt{5}\,dx\,dy$

$$\iint_S x^2 z\,dS = \iint_D x^2\cdot\sqrt{x^2+y^2}\cdot\sqrt{5}\,dx\,dy = \left| \begin{matrix} x = \rho\cos\varphi \\ y = \rho\sin\varphi \\ \frac{d(x,y)}{d(\rho,\varphi)} = \rho \end{matrix} \right| =$$

$$= \int_0^{2\pi}\int_1^2 \rho^2\cos^2\varphi\cdot\rho\cdot\sqrt{5}\cdot\rho\,d\rho\,d\varphi = /\ldots/ = \frac{62\sqrt{5}}{5}\pi$$

**Svar:** $\frac{62\sqrt{5}}{5}\pi$

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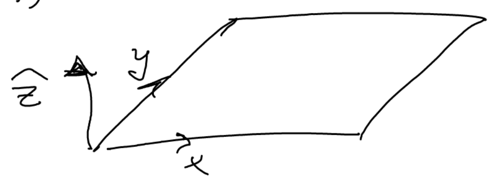
(2) $S: \; z = \rho^2\sin 2\varphi$, $(\rho, \varphi, z) = (1, \frac{\pi}{4}, 1)$




- Obs, $z = 1^2\sin(2\cdot\frac{\pi}{4}) = 1$ ok.
- Normal till xy-planet är $\hat{z}$
- Normal till $S$ fås ur implicit ekvationen:

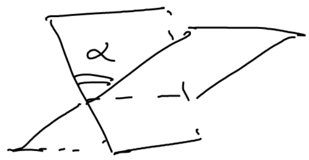
$S = \{(\rho, \varphi, z) : f(\rho, \varphi, z) := \rho^2\sin 2\varphi - z = 0\}$, d.v.s.

$\bar{n} = \text{grad } f = /$ i cylindriska koordinater $/ = \frac{\partial f}{\partial \rho}\hat{\rho} + \frac{1}{\rho}\frac{\partial f}{\partial \varphi}\hat{\varphi} + \frac{\partial f}{\partial z}\hat{z} =$

$$= 2\rho\sin 2\varphi\cdot\hat{\rho} + \frac{2\rho^2\cos 2\varphi}{\rho}\,\hat{\varphi} - \hat{z} = \Big/ \text{ i punkten } \left(1,\tfrac{\pi}{4},1\right) \Big/ =$$

$$= 2\hat{\rho} - \hat{z}.$$

Observera att vinkel mellan två plan ligger mellan 0 och $\frac{\pi}{2}$ $\Rightarrow$




$$\cos\alpha = \left|\frac{\bar{n}\cdot\hat{z}}{|\bar{n}|\cdot|\hat{z}|}\right| = \left|\frac{-1}{\sqrt{5}}\right| = \frac{1}{\sqrt{5}}$$

<u>Svar:</u> $\alpha = \arccos\frac{1}{\sqrt{5}}$

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3) $\bar{A} = \frac{x}{x^2+y^2}\,\hat{x} + \frac{y}{x^2+y^2}\,\hat{y}$, $\quad \bar{B} = -\frac{y}{x^2+y^2}\,\hat{x} + \frac{x}{x^2+y^2}\,\hat{y}$

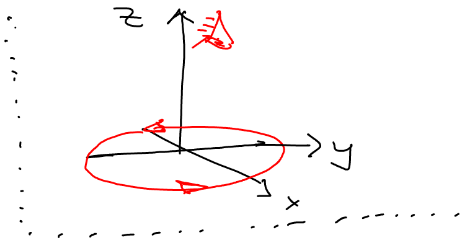
a) $L:\ x^2+y^2=4,\ z=0$ $\quad : \bar{r}\ \begin{cases} x = 2\cos t \\ y = 2\sin t \\ z = 0 \end{cases}$ $\quad t: 0 \to 2\pi$

(<u>orientering moturs</u>)

$\bar{r} = 2\cos t\cdot\hat{x} + 2\sin t\cdot\hat{y}$

$d\bar{r} = -2\sin t\,\hat{x} + 2\cos t\,\hat{y}$

$\bar{A} = \frac{\cos t}{2}\,\hat{x} + \frac{\sin t}{2}\,\hat{y}$, $\quad \hat{B} = -\frac{\sin t}{2}\,\hat{x} + \frac{\cos t}{2}\,\hat{y}$




$$I_1 = \oint_L \bar{A}\cdot d\bar{r} = \int_0^{2\pi}\left(-\cos t\sin t + \sin t\cos t\right)dt = 0$$

$$I_2 = \oint_L \bar{B}\cdot d\bar{r} = \int_0^{2\pi}\left((\sin t)^2 + (\cos t)^2\right)dt = 2\pi$$

b) $\bar{B}$ är $\boxed{\text{ej}}$ potentiellt fält (ty $I_2 \neq 0$), d.v.s. integralen $\oint_L \bar{B}\cdot d\bar{r} \neq 0$ längs en <u>sluten</u> kurva

$\bar{A}$ : söker en potential i D


$$\varphi'_x = \frac{x}{x^2+y^2}, \quad \varphi'_y = \frac{y}{x^2+y^2}, \quad \varphi'_z = 0$$

$$\Rightarrow \varphi = \varphi(x,y) = \int\frac{x\,dx}{x^2+y^2} = \frac{1}{2}\int\frac{d(x^2+y^2)}{x^2+y^2} = \frac{1}{2}\ln(x^2+y^2) + g(y)$$

$$\varphi'_y = \frac{y}{x^2+y^2} + g'(y) \Leftrightarrow \frac{y}{x^2+y^2} \Rightarrow g'(y) = 0 \Rightarrow g = C = \text{konst.}$$

$\Rightarrow \varphi = \frac{1}{2}\ln(x^2+y^2) + C$ är en potential för $\bar{A}$ i $D = \{x^2+y^2 \neq 0\}$.

Svar a) se ovan, b) $\bar{A}$ är ett potentiellt fält, $\bar{A} = \nabla \varphi$

$\bar{B}$ är ej potentiellt fält

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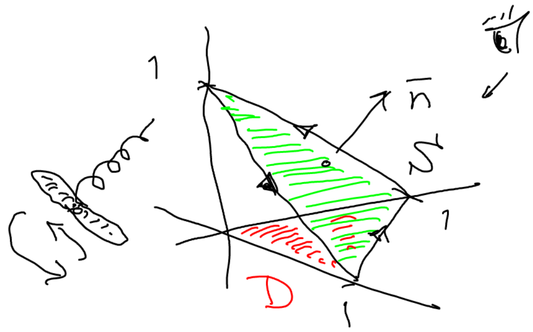
(4) $\bar{A}(x,y,z) = y^2\hat{x} + x^2\hat{z}$




Stokes:

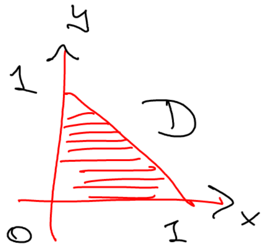
$$\oint_\Gamma \bar{A}\cdot d\bar{r} = \iint_S \operatorname{rot}\bar{A}\cdot\hat{n}\, dS$$

- Orientering av $\Gamma$ (se bilden) medför att normalen i Stokes satsen skall vara uppåtriktad ( S ska ligga till vänster om man går längs $\Gamma$ ).
- $(1,0,0)$, $(0,1,0)$, $(0,0,1)$ definierar planet $S: x+y+z=1$

$\Rightarrow$ $z = 1-x-y$, $\bar{n} = (-z'_x, -z'_y, 1) = (+1,+1,1) = (1,1,1)$

Obs. uppåt!

- $\operatorname{rot}\bar{A} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial_x & \partial_y & \partial_z \\ y^2 & 0 & x^2 \end{vmatrix} = (0; -2x; -2y)$




- $d\bar{S} = \hat{n}\, dS = (1,1,1)\, dx\,dy$
- $\iint_S \operatorname{rot}\bar{A}\cdot d\bar{S} = \iint_D -(2x+2y)\,dx\,dy =$

$$= -2\int_0^1\int_0^{1-x}(x+y)\,dy\,dx = -2\int_0^1 \left[xy + \frac{y^2}{2}\right]_0^{1-x} = -2\int_0^1 \frac{1-x^2}{2}\,dx = -\frac{2}{3}$$

Svar $-\frac{2}{3}$

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(5) $\bar{A}(x,y,z) = (\hat{x}\cdot\bar{r})f(r)\,\bar{r}$, $r = |\bar{r}|$.

Obs.: $\hat{x}\cdot\bar{r} = x$ $\quad \nabla x = \hat{x}$, $\nabla r = \hat{r}$, $\nabla f(r) = f'(r)\nabla r = f'(r)\,\hat{r}$

$$\operatorname{div}\bar{A} = \operatorname{div}(x\cdot f(r)\cdot\bar{r}) = \nabla(xf(r))\cdot\bar{r} + xf(r)\operatorname{div}\bar{r} =$$

$$= (x\nabla f(r) + f(r)\nabla x)\cdot\bar{r} + xf(r)\cdot 3 = xf'(r)\,\hat{r}\cdot\bar{r} + f(r)\,\hat{x}\cdot\bar{r} + 3xf(r) =$$

$$= rxf'(r) + xf(r) + 3xf(r)$$

vilket ger att

$$x(r f'(r) + 4 f(r)) = 0 \Rightarrow r f'(r) + 4 f(r) = 0 \quad \text{för } x \neq 0.$$

Integrerande faktor är $\exp \int \left( \frac{4}{r} dr \right) = \exp 4 \ln r = r^4$

$$\Rightarrow r^4 f'(r) + 4r^3 f(r) = (r^4 f(r))' = 0 \Rightarrow f(r) = \frac{C}{r^4}.$$

$$A(1,0,0) = 2\hat{x} \Rightarrow A(1,0,0) = 1 \cdot f(1) \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix} \Rightarrow f(1) = 2 \Rightarrow C = 2$$

Svar: $f(r) = \frac{2}{r^4}$

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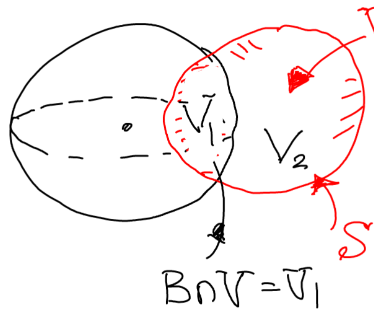
(6) $\bar{F}(x,y,z) = (4x - x^3)\hat{x} + (4y - y^3)\hat{y} + (4z - z^3)\hat{z}$

Flödesintegral över en sluten yta $S$; m.h.a. Gausssatsen

$$I = \iint_S \bar{F} \cdot d\bar{S} = \iiint_V \operatorname{div} \bar{F} \cdot dV = \iiint_V (4+4+4 - 3x^2 - 3y^2 - 3z^2)\, dV =$$

$$= 3 \iiint_V (4 - (x^2+y^2+z^2))\, dV$$

$B : x^2+y^2+z^2 \leq 4$




där $V$ är kroppen vars rand $= S$

Observera att integranden är icke-negativ exakt då $4 - (x^2+y^2+z^2) \geq 0$ om och endast om $(x,y,z) \in V_1 = B \cap V$, medan $4 - (x^2+y^2+z^2) < 0$ på $V_2 = V \setminus V_1$, vilket ger

$$I = 3 \iiint_{V_1} \underbrace{(4 - x^2 - y^2 - z^2)}_{+} dV + 3 \iiint_{V_2} \underbrace{(4 - x^2 - y^2 - z^2)}_{-} dV \leq 3 \iiint_{V_1} (4 - x^2 - y^2 - z^2)\, dV \leq$$

$$\leq 3 \iiint_B (4 - r^2)\, dV = 3 \int_0^{2\pi} \int_0^{\pi} \int_0^{2} (4 - r^2)\, r^2 \sin\theta \, dr\, d\theta\, d\varphi = \frac{256}{5}\pi$$

(I fall om $V_2 \neq \emptyset$ så blir $I < \frac{256}{5}\pi$ (eftersom $-$-integralen ovan blir negativ!). Detta bevisar att det största (positiva) värdet som $I$ antar blir $\frac{256}{5}\pi$ exakt då $V$ ges av $x^2+y^2+z^2 \leq 4$, alltså $S$ ges av $x^2+y^2+z^2 = 4$.

Svar: $S : x^2+y^2+z^2 = 4$ och $I_{max} = \frac{256}{5}\pi$.