Exercises on Hyperbolic Plane

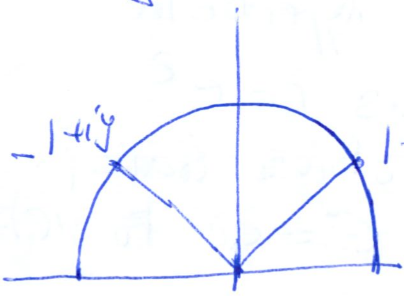
15.2.2 Consider the function $\sinh z = \frac{e^{z}-e^{-z}}{2}$ and $\cosh(z) = \frac{e^{z}+e^{-z}}{2}$

a) $\cosh^2 z - \sinh^2 z = \frac{e^{2z}+e^{-2z}-e^{2z}-e^{-2z}+2+2}{4} = 1$

b) $\cosh(2z) = \frac{e^{2z}+e^{-2z}}{2} = 2\left(\frac{e^{2z}+e^{-2z}+2-2}{4}\right) = 2\cosh^2 z - 1$

$$= 2\left(\frac{e^{2z}+e^{-2z}-2+2}{4}\right) = 2\sinh^2 z + 1$$

15.2.3 Find the hyperbolic distance between $z = 1+iy$ and $w = -1+iy$ as a function of y.
Show that for a given positive t there is a value of y s.t $\rho(1+iy, -1+iy) = t$.



We have that

$$\cosh^2 \tfrac{1}{2}\rho(z,w) = \frac{|1+yi-(-1+iy)|^2}{4y^2}$$

$$\cosh^2 \tfrac{1}{2}\rho(z,w) = \frac{1+y^2}{y^2}$$

By 15.2.2b) $\cosh\rho(z,w) = 2\cosh^2\tfrac{1}{2}\rho(z,w) - 1$

$$= \frac{2+2y^2}{y^2} - 1 = \frac{2+y^2}{y^2}, \quad y>0$$

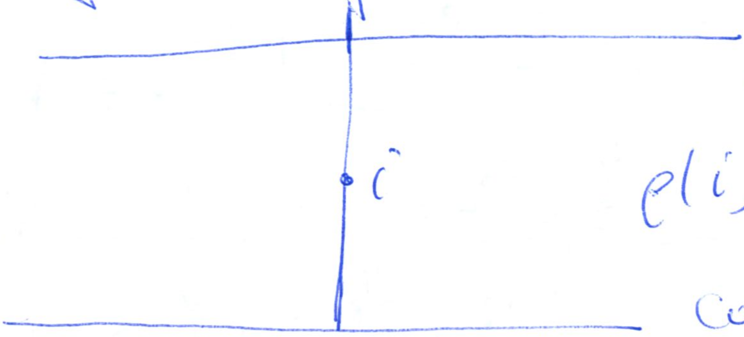
Now given $t>0$ s.t $\frac{e^{2t}+1}{2e^{t}} = \frac{2+y^2}{y^2}$, $y>0$

$s=e^{t}$ satisfies $y^2 s^2 - 2(2+y^2)s + y^2 = 0$

or $y^2(e^{t}-1)^2 = 4e^{t}$

so $y = \frac{2e^{\frac{1}{2}t}}{e^{t}-1}$ (Observe that $t>0$, $e^{t}>1$)

15.2.4 Consider the set $L \subseteq \mathcal{H}$ defined by the equation $L = \{ z \in \mathbb{C} ; \ \operatorname{Im}[z] = 2 \}$



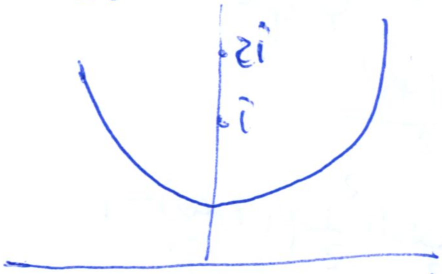
The distance $\rho(i, x+2i)$ satisfies that

$$\cosh \frac{1}{2} \rho(i, x+2i) = \frac{|i-x+2i|}{8} = \frac{x^2+9}{8}$$

As $\cosh \frac{1}{2}\rho(i, x+2i)$ has the minimum for $x=0$ the distance $\rho(i, x+2i)$ has the minimum for $x=0$ i.e. $\underline{z = 2i}$ So

$$\rho(i, L) = \rho(i, 2i)$$

15.3.1 Give the equation of the hyperbolic circle with centre $2i$ and radius $r = e^2$



First of all the Mobius transf. $g(z) = \frac{1}{2} z$ takes $C = 2i$ to $g(C) = i$ and $g^{-1}(z) = 2z$

Now, we know that the equation for $g(C)$ is $x'^2 + (y' - \cosh e^2)^2 = \sinh^2(e^2)$ where

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 1/2 & 0 \\ 0 & 1/2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

So $C = g^{-1}(gC)$ has the equation $\frac{1}{4} x^2 + \frac{1}{4}(y - 2\cosh e^2)^2 = \sinh^2(e^2)$

$$x^2 + (y - 2\cosh e^2)^2 = (2\sinh(e^2))^2$$

For $x = 0$ $\quad (y - 2\cosh e^2)^2 = \begin{cases} 2\sinh(e^2) \\ -2\sinh(e^2) \end{cases}$

$z_1 = 2(\cosh e^2 + \sinh e^2) i$

$z_2 = 2(\cosh e^2 - \sinh e^2) i$

15.3.2 We compare the equations for circles in Euclidean geometry, Spherical, Hyperbolic

E G

$(x-a)^2+(y-b)^2=r^2$

$\boxed{x^2+y^2=r^2}$

S G

$(\cos\lambda\cos\theta, \cos\lambda\sin\theta, \sin\lambda)$

(λ, θ) at distance $\frac{\pi}{2}-\lambda$ to $N(0,0,1)$

$\frac{\pi}{2}-\lambda = r$

So $x^2+y^2+\cos^2 r=1$

or $\boxed{x^2+y^2=\sin^2 r}$

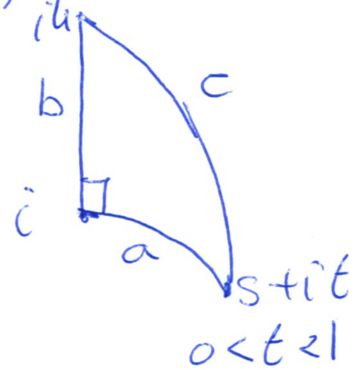
H G

centre i, $z=x+iy$, $y>0$

$\boxed{x^2+(y-\cosh r)^2=\sinh^2 r}$

<u>15.4.1</u> Suppose that a, b, c are the sides of a right-angled triangle with the right-angle opposite the side of length c. Show that $c \leq a+b$.

First of all, we can assume that the triangle is



We know that

$$\cosh a = \frac{k^2+1}{k^2}$$

$$\cosh b = \frac{1}{t}$$

$$\cosh c = \frac{k^2+1}{2tk}$$

We know also that $\cosh(a+b)=\cosh a\cosh b+\sinh a\sinh b$

We see that $\cosh c \leq \cosh(a+b)$.

As $\cosh(x)$ is increasing on positive real numbers then $c \leq a+b$