Hand-in Exercises 3 i TATA49 Geometry with Applications

In the following the coordinates are homogeneous if nothing else is said. Calculations can be done with computer. As usual we use your birthday parameters. However, if someone is born October 20, November 22 or December 24, the day of birth for our purposes is 23.

Observe that the only distinction in this sheet between points and lines is that points will use $(::)$ and lines $[::]$. But both are homogeneous coordinates

Exercise 1 *Determine the projection of the cube with vertices* $A_1(-1,-1,-1,1)$, $A_2(1,-1,-1,1), A_3(-1,1,-1,1)$, $A_4(1,1,-1,1)$, $A_5(-1,-1,1,1)$, $A_6(1,-1,1,1)$, $A_7(-1,1,1,1)$, $A_8(1,1,1,1)$ *on the plane* $x_1 + 2x_2 - 2x_3 - 5(x_4) = 0$ *from the (vanishing) point* $V(1/8, 1/4, 0, 1)$. *The coordinate system for the plane is* $O(r_1, 3r_1, r_1, r_1), X = (2,1,2,0)$ *and* $Y = (2,-2,-1,0)$.

Exercise 2 *Consider the line* $l[r_1 : r_2 : p_1+p_2]$. *Show that the point* $X(1:0:0)$ *does not belong to* l. *Determine the coordinates of the image of the points in* l *under the perspectivity from* l *to* $m[1:1:0]$ *with centre* X

Exercise 3 *Let* l, m *be the lines in Exercise 2. Give the matrix of the perspectivity in Exercise 2 using as base points: the points* $P(p_1+p_2 : 0 : -r_1), Q(0 : p_1+p_2 : -r_2)$ *for* l*; and the points* $R(1:-1:0), Z(0:0:1)$ *for* m.

Exercise 4 *Consider collineations that fix all the lines* $m_\alpha[\cos(\alpha) : \sin(\alpha) : r_1\cos(\alpha) + r_2\sin(\alpha)]$ *as well as* $m[r_1 : r_2 : 1]$. *Show that they are homologies depending on one (1) parameter. Determine their centre(s), axis(axes) and the geometrical meaning of the parameter.*

Exercise 5 *A projective line has affine part the affine line with equation* $(x_1, x_2) = (r_1 - 3, r_2 + 5) + t(2(q_1+3), -(q_2+1))$. *Determine the point at infinity of* l *and the point* $D(d_1 : d_2 : d_3)$ *on the line such that the cross ratio of the points* A, B, C *with real parameters* $t = 1, 2, 3$ *respectively and* D *is* $-1/2$. *What is the real parameter for* D*? (Notice that the real parameter for* D *can be* ∞*).*

Exercise 6 *Determine the collineation that sends* $P(p_1 - 2 : p_2 + 1 : 1)$ *to* $P'(1:1:0)$*;* $Q(q_1+1 : q_2+1 : 1)$ *to* $Q'(0:1:1)$*;* $R(r_1 - 3 : r_2 + 1 : 1)$ *to* $R'(1:0:1)$ *and finally* $S(p_1 + r_1 : q_2 + r_2 : 2)$ *till* $U(1:1:1)$. *Determine its fixed elements, make a diagram of the invariant elements.*

Exercise 7 *Determine the collineation that sends the lines:* $x[1:1:0]$ *to* $q[q_1 - 2 : q_2 + 1 : 1]$, $y[0:1:1]$ *to* $p[p_1+3 : p_2+3 : 1]$*;* $z[1:0:1]$ *to* $s[p_1 + r_1 : q_2 + r_2 : 3]$ *and finally* $u[1:1:1]$ *to* $r[r_1 - 3 : r_2 + 1 : 1]$. *Determine the fixed elements. Determine the diaognal lines of the complete quadrilateral* $qpsr$.

Exercise 8 *Determine the invariant elements under the collineation with matrix* $\begin{pmatrix} -r_1^2 + r_2^2 & 2r_1^2 + 2r_1r_2 & -2r_1r_2 \\ 0 & a & 0 \\ -2r_1r_2 & 2r_2^2 + 2r_1r_2 & r_1^2 - r_2^2 \end{pmatrix}$. *For which values of a is the collineation perspective? In that case is it a homology or an elation? In case the collineation is an homology, determine the cross-ratio of the homology.*

Exercise 9 *Determine the polarity that sends the point* $X(1:0:0)$ *to the line* $z[0:0:1]$; *the point* $Z(0:0:1)$ *to the line* $y[0:1:0]$; $R(r_1:r_2:0)$ *to the line* $r[r_1:r_2:0]$ *and the point* $Q(r_2:-r_1:1)$ *to the line* $q[3r_2:-3r_1:1]$. *Give the equation for the conic, and for the tangents to the conic. Is any of the points or lines above self-conjugate?*

Alternative Exercise 9: *Determine the polarity that sends the point* $X(1:0:1)$ *to the line* $z[0:0:1]$; *the point* $Z(0:0:1)$ *to the line* $y[1:1:1]$ *and* $R(r_1:r_2:0)$ *to the line* $r-1:r_2:1]$. *Give the equation for the conic, and for the tangents to the conic. Is any of the points or lines above self-conjugate?*

Exercise 10 *Determine the polarities for which the triangle* PQR, *with* $P(p_1:p_2:1)$, $Q(q_2:0:q_1)$, $R(0:r_2:r_1)$, *is self-polar.*

Exercise 11 *Let* T *be a collineation. Show that if* m *is the polar line to a point* P *with respect to some conic* $\mathcal{C}$ *with matrix* C, *then* $m' = T(m)$ *is the polar to the point* $P' = T(P)$ *with respect to a conic* $\mathcal{C}'$ *with matrix* C'. *Control first that* $\mathcal{C}'$ *is a conic.*

Exercise 12 *Let* T *be a homology whose centre* $Z(0:0:1)$ *and axis* $z[0:0:1]$ *are pole-polar with respect to a polarity with (symmetric) matrix* C *and conic* $\mathcal{C}$. *Caculate the matrix of* T. *Show that* T *fixes the polarity and so the conic* $\mathcal{C}$.

Exercise 13 *A projective point* P *is singular with respect to a polarity with matrix* C *if* $P^tCX = 0$ *for every projective point* X. *Show that if* P *and* Q *are singular points the whole line* PQ *is a line of singular points, and so belongs to the conic of the polariity.*

Exercise 14 *Determine the coordinates of the points in a projective plane of order 5. Use triples of the symbols 0, 1, 2, 3 and 4 and arithmetic modulus 5. Determine every line of the plane. Once you have chosen the line at infinity, classify the affine lines according to wether they are parallel or not. Notice that the coordinates* $(1:0:1); (2:0:2); (3:0:3); (4:0:4)$ *represent the same point.*