Hand-in Exercises 2, TATA49 Geometry with Applications

Maple or Matlab may be used in the calculations.

The exercises are customised by your birthdays as in the previous sheet of exercises.

Exercise 1 *Do your computer aided portrait with instances of a quadrangle with vertices* $O(0,0)$, $R_1(q_1+r_1+5r_2,0)$, $R_3(q_1+r_1,5r_2)$, $R_2(2q_1+2r_1+5r_2,5r_2)$ *. First of all transform the quadrangle such that the side has length 50. Now, the mouth, centered on width has ratios 1:15 (width) and 1:50 (height) to the face, the height of the mouth divides the height of the face in ratio 1:9 from the bottom. The two eyes lie (also symmetric in width) on a height 3:4 to the face height; they have ratio 1:20 to the face (in both measures). Which affine transformations do you have to do? Give their matrices.*

Exercise 2 *Determine the images under the stereographic projection* $\varphi : \widehat{\mathbb{C}} \to \mathbb{S}^2$ *of all the circles in* $\mathbb{C}$ *with equations* $x^2+y^2-2ay+a^2-1=0$ *, with* $a \in \mathbb{R}$ *a parameter.*

Exercise 3 *Use spherical coordinates* $\mathbf{x}(\lambda,\varphi) = (\cos(\lambda)\cos(\varphi), \sin(\lambda)\cos(\varphi), \sin(\varphi))$ *, where* λ *is the longitude and* φ *is the latitude to provide a parametrization of the circles obtained in the previous exercise using as parameter the latitude* φ *.*

Exercise 4 *Determine the images under the inverse of the stereographic projection* $\phi : \mathbb{S}^2 \to \widehat{\mathbb{C}}$ *of all the great circles in* $\mathbb{S}^2$ *determined by the planes* $cx_2 - \sqrt{1-c^2}x_3 = 0$ *, with* $-\pi/2 < c < \pi/2$ *a parameter.*

Exercise 5 *Calculate the distances of the great circles in the family in Exercise 4 to the North Pole* N *.*

Exercise 6 ***a)*** *Calculate the angles in the spherical triangle with vertices A (longitude* $(r_2{+}3)(p_1{+}p_2)$ *deg, latitude 0 deg), B (longitude 0 deg, latitude* $q_2q_1{+}45$ *deg) and C (longitude* $(r_1{-}2)p_2$ *deg, latitude 30 deg). Observation* $(r_2+3)(p_1+p_2)$ *is a number with two or three digits:* p_1+p_2 *in the units and* r_2+3 *in the tens, and possible hundreds. In the same way* q_2q_1 *is a number with two digits:* q_1 *in the units and* q_2 *in the tens, and* $(r_1-2)p_2$ *is a number with two digits:* p_2 *in the units and* r_1-2 *in the tens.*

b) *Calculate the area of the triangle if we consider that the triangle is on Earth (Radius of Earth is 6371 km).*

Exercise 7 ***a)*** *Determine the great circles that support the sides of the spherical triangle in exercise 6 and their images under the inverse of the stereographic projection.*

b) *Determine the sizes of the edges of the spherical triangle (on Earth) in exercise 6 (Radius of Earth is 6371 km).*

Exercise 8 *Consider* $\mathbb{R}^4$ *with coordinates* $\mathbf{v} = (v_1,v_2,v_3,v_0)$ *(Observe that the real part of the quaternion is the fourth coordinate). Given the quaternion* $q_0 = a_1\mathbf{i}+a_2\mathbf{j}+a_3\mathbf{k}+a_0 = ((a_1,a_2,a_3),a_0)$ *, show that the multiplication with* q_0 *to the left is a linear transformation in* $\mathbb{R}^4$ *with matrix*

$$\begin{pmatrix} a_0 & -a_3 & a_2 & a_1 \\ a_3 & a_0 & -a_1 & a_2 \\ -a_2 & a_1 & a_0 & a_3 \\ -a_1 & -a_2 & -a_3 & a_0 \end{pmatrix}.$$

Calculate the matrix for the multiplication with q_0 *on the right.*

Exercise 9

1. *Use Exercise 8 to determine the matrix of the rotation in* $\mathbb{A}^3$ *with quaternion* $q_0 = a_1\mathbf{i}+a_2\mathbf{j}+a_3\mathbf{k}+a_0$, $|q_0| = 1$ *.*
2. *Use Exercise 8 to determine the matrix of the reflection in* $\mathbb{A}^3$ *on the plane with equation* $8x_1+4x_2-8x_3=0$ *. Observe that the last row and column of the matrix equal* $(0,0,0,1)$ *.*

Exercise 10 *Consider an animation of an object with start orientation (time $t = 0$) given by the quaternion $q_s = (r_1+p_1, (r_2+p_2, p_1+q_2, q_1+p_2))/l$ where $l = \sqrt{(q_1 + p_2)^2 + (p_1 + q_2)^2 + (r_2 + p_2)^2 + (r_1 + p_1)^2}$ and final orientation (time $t = 1$) given by the quaternion $q_f = (1/2, (-1/2, 1/2, -1/2))$. Give the orientations at times $t = (2j - 1)/20,\ j = 0, 1, 2, \ldots, 10$.*
Use Exercise 9.1 to give the matrices of the rotations corresponding to the orientations for times $t_1 = (p_1 + q_2 + r_2)/20$ and $t_1 = (3p_1 + 3p_2)/20$

Exercise 11 *Identify the isometry with matrix:*
$$\begin{pmatrix} -1/3 & (1-\sqrt{3})/3 & (-1-\sqrt{3})/3 & -1/3 \\ (1+\sqrt{3})/3 & -1/3 & (1-\sqrt{3})/3 & -2/3 \\ (-1+\sqrt{3})/3 & (1+\sqrt{3})/3 & -1/3 & 2/3 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$
. Does the isometry fix some plane? Which? Does it fix any point or line?

Exercise 12 *Consider the 3D-rotation given by the quaternion $(\cos(r_2\pi/20), \sin(r_2\pi/20)(0, 0, 1))$. Show that this rotation is a rotation f of $\mathbb{S}^2$. Show that the transformation $F = \varphi^{-1} \cdot f \cdot \varphi$ is a rotation of the (extended) plane. Determine its centre and angle.*

Exercise 13 *Consider a computer animation consisting only of translations. First translation T_0 has vector $\mathbf{v}_0$ and last translation T_1 has vector $\mathbf{v}_1$. Show that any intermediate translation $T(t)$, $0 \le t \le 1$, in the animation can be written $T(t) = (1-t)T_0 + tT_1$. Give the matrix of any intermediate translation.*

Exercise 14 *Describe the symmetries of the friezes below:*