Hand-in Exercises 3 i TATA49 Geometry with Applications

In the following the coordinates are homogeneous if nothing else is said. Calculations can be done with computer. As usual we use your birthday parameters as in the previous asingnments. However, if someone is born October 20, November 22 or December 24, the day of birth for our purposes is 23.

Observe that the only distinction in this sheet between points and lines is that points will use (::) and lines [::]. But both are homogeneous coordinates

Exercise 1 *Determine the plane coordinates of the projection of the octohedron with vertices with affine coordinates* $A_1(0,0,1,1)$, $A_2(0,0,-1,1)$, $A_3(-1,0,0,1)$, $A_4(1,0,0,1)$, $A_5(0,-1,0,1)$, $A_6(0,1,0,1)$ *on the plane with coordinate system* $O(2r_2, r_2, r_2, 1), X = (6,2,-3,0)$ *and* $Y = (2,-3,6,0)$ *from the (vanishing) point* $V(8,4,8,1)$. *Give also an equation for the projection plane.*

Exercise 2 *Consider the line* $l[r_1 : r_2 : p_1+p_2]$. *Show that the point* $Y(0:1:0)$ *does not belong to* l. *Determine the coordinates of the image of the points in* l *under the perspectivity from* l *to* $m[q_1 : q_1 + q_2 : 0]$ *with centre* Y. *Observe that a projective line consists of an affine line plus its point at infinity.*

Exercise 3 *Let* l, m *be the lines in Exercise 2. Give the matrix of the perspectivity in Exercise 2 using as base points: the points* $P(p_1 + p_2 : 0 : -r_1), Q(0 : p_1 + p_2 : -r_2)$ *for* l*; and the points* $R(q_1 + q_2 : -q_1 : 0), Z(0:0:1)$ *for* m.

Exercise 4 *Consider elations with axis* $z[0:0:1]$ *and centre* $C(r_2 : q_1+q_2 : 0)$. *Show that they are affinities depending on one (1) parameter. Identify these affinities and the geometrical meaning of the parameter.*

Exercise 5 *Show that the points* $A(r_2, p_2, q_2)$, $B(q_1, r_1, p_1)$, $C(r_1r_2+q_1^2, p_2r_1+q_1r_1, q_2r_1+p_1q_1)$*are collinear. Determine the point* D *collinear with* A, B, C *such that* $R(A, D, C, B) = 1/2$. *As any line in the real projective plane is a line of real numbers plus its point at infinity, determine the real coordinates of* A, B, C, D *(Which point has you chosen as origin of the line?).*

Exercise 6 *An involution* T *is a transformation of order 2, i.e.* $T \cdot T = 1_d$.

1. *Show that a projectivity that is an involution (not the identity) has a matrix* $\begin{pmatrix} a & b \\ c & -a \end{pmatrix}$, *with* $a^2 + bc \neq 0$.
2. *Show that a projectivity on the line* l *that interchanges two points* P, Q *on* l *is an involution. Use an appropriate basis.*

Exercise 7 *Determine the collineation that sends* $x[1:0:0]$ *to* $p[p_1 : p_2 : 1]$, $y[0:1:0]$ *to* $q[q_1 : q_2 : 1]$, $z[0,0,1]$ *to* $r[r_1 : r_2+2 : 0]$ *och* $u[1:1,1]$ *to* $s[p_1+q_1+r_1 : p_2+q_2+r_2+2 : 2]$. *Determine its fixed elements (points and lines).*

Exercise 8 *Consider the affine plane as a part of the projective plane. Consider the affine line m with equation $-\sin\theta x_1 + \cos\theta x_2 = 0$ (and homogenous coordinates $m[-\sin\theta, \cos\theta, 0]$). Show that the reflection in the line m is a homologi. Determine its centre, axis and cross-ratio.*

Exercise 9 *Determine the invariant elements under the collineation with matrix* $\begin{pmatrix} -r_1^2 + r_2^2 & 2r_1^2 + 2r_1r_2 & -2r_1r_2 \\ 0 & a & 0 \\ -2r_1r_2 & 2r_2^2 + 2r_1r_2 & r_1^2 - r_2^2 \end{pmatrix}$. *For which values of a is the collineation perspective? In that case is it a homology or an elation? In case the collineation is a homology, determine the cross-ratio of the homology.*

Exercise 10 *Give a polarity for which the triangle PQR, with $P(p_1{+}2 : p_2 : 1)$, $Q(q_2 : q_1 : 0)$, $R(0 : r_2 : r_1)$, is self-polar. Determine the conic associated to that polarity. which kind of polarity is it?*

Exercise 11 *Let T be a collineation. Show that if m is the polar line to a point P with respect to some conic $\mathcal{C}$ with matrix C, then $m' = T(m)$ is the polar to the point $P' = T(P)$ with respect to a conic $\mathcal{C}'$ with matrix C'. Control first that $\mathcal{C}'$ is a conic.*

Exercise 12 *Let T be a collineation with orthogonal matrix ($A_T^{-1} = A_T^t$). Show that T leaves the conic associated to an elliptic polarity invariant. You can assume that the triangle $X(1:0:1), Y(0:1:0), Z(0:0:1)$ is self-polar with respect to the polarity.*

Exercise 13 *A projective point P is singular with respect to a polarity with matrix C if $P^tCX = 0$ for every projective point X. Show that if P and Q are singular points the whole line PQ is a line of singular points, and so belongs to the conic of the polariity.*

Exercise 14 *Determine the coordinates of the points in a projective plane of order 4. Use $\mathbb{F}_4 = \{0, 1, 2, 3\}$ with the usual addition modulo 4 and the following rules for multiplication: $0 \cdot x = 0$, $1 \cdot x = x$, $2 \cdot 2 = 3$, $2 \cdot 3 = 1$, $3 \cdot 3 = 2$. Determine every line of the plane. Once you have chosen the line at infinity, classify the affine lines according to wether they are parallel or not. Notice that the coordinates $(1:2:0); (2:3:0); (3:2:0)$ represent the same point. Use your previous result to give three POLS of order 4.*