

Hand-in Exercises 2, TATA49 Geometry with Applications. Fall 2021

Maple or Matlab may be used in the calculations.

The exercises are customized by your birthdays as in the previous sheet of exercises.

Exercise 1 Determine the images under the inverse of the stereographic projection $\phi : \mathbb{S}^2 \rightarrow \widehat{\mathbb{C}}$ of all the circles in \mathbb{C} passing $A(1, 0, 0)$, $B(-1, 0, 0)$ and $S_a(0, \cos a, \sin a)$, with $\frac{\pi}{q_2+5} \leq a < \frac{\pi}{2}$ a parameter. What are the longitude and latitude of S_a ?

Exercise 2 1. Calculate the distances from $N(0, 0, 1)$ to the circle S_a .

2. Using spherical coordinates $(\cos \lambda \cos \theta, \cos \lambda \sin \theta, \sin \lambda)$ give equations of the circles S_a parametrised by the latitude λ .

Exercise 3 Determine the images under the stereographic projection $\varphi : \widehat{\mathbb{C}} \rightarrow \mathbb{S}^2$ of all the lines l_t in \mathbb{C} with equations $x(p_1 + p_2 + t) + y(q_1 + q_2 - t) - 2t = 0$, with t a parameter.

Exercise 4 Determine the planes to which the circles in Exercise 3 belong, and the angles formed, pairwise, by the circles for $t_1 = 1, t_2 = -1, t_3 = 1/2$ at $N(0, 0, 1)$

Exercise 5 1. Calculate the angles in the spherical triangle with vertices $A(\text{longitude } (r_1 - 3)r_2 \text{ deg, latitude } 0 \text{ deg})$, $B(\text{longitude } (r_1 - 3)r_2 \text{ deg, latitude } 60 \text{ deg})$ and $C(\text{longitude } 0 \text{ deg, latitude } (r_1 - 3)r_2 \text{ deg})$. Observation $(r_1 - 3)r_2$ is a number with two digits: r_2 in the units and $r_1 - 3$ in the tens. Observe that the calculations are done in radians.

2. Calculate the area of the triangle if we consider that the triangle is on Earth (Radius of Earth is 6371 km).

Exercise 6 1. Determine the images under the inverse of the stereographic projection of the great circles in Exercise 5.

2. Determine the sizes of the edges of the spherical triangle (on Earth) in exercise 5 (Radius of Earth is 6371 km).

Exercise 7 Consider \mathbb{R}^4 with coordinates $\mathbf{v} = (v_1, v_2, v_3, v_0)$ (Observe that the real part of the quaternion is the fourth coordinate). Given the quaternion $q_0 = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k} + a_0 = ((a_1, a_2, a_3), a_0)$, show that the multiplication with q_0 to the

left is a linear transformation in \mathbb{R}^4 with matrix
$$\begin{pmatrix} a_0 & -a_3 & a_2 & a_1 \\ a_3 & a_0 & -a_1 & a_2 \\ -a_2 & a_1 & a_0 & a_3 \\ -a_1 & -a_2 & -a_3 & a_0 \end{pmatrix}.$$

Calculate the matrix for the multiplication with q_0 on the right.

Exercise 8 1. Use Exercise 7 to determine the matrix of the rotation in \mathbb{E}^3 with quaternion $q_0 = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k} + a_0$, $|q_0| = 1$.

2. Use Exercise 7 to determine the matrix of the reflection in \mathbb{E}^3 on the plane with equation $\frac{x_1}{3} + \frac{2x_2}{3} - \frac{2x_3}{3} + 0(1) = 0$. Observe that the last row and column of the matrix equal $(0, 0, 0, 1)$.

Exercise 9 Consider an animation of an object with start orientation (time $t = 0$) given by the quaternion $q_s = (p_1 + 1, q_1 + 2, r_1 - 3, r_2 + 1)/l$ where $l = \sqrt{(p_1 + 1)^2 + (q_1 + 2)^2 + (r_1 - 3)^2 + (r_2 + 1)^2}$ and final orientation (time $t = 1$) given by the quaternion $q_f = (6/9, (5/9, 4/9, 2/9))$. Give the orientations at times $t = j/10$, $j = 0, 1, 2, \dots, 10$

Exercise 10 Identify and describe completely the isometry of \mathbb{E}^3 with matrix:

$$\begin{pmatrix} 1/3 & 2/3 & 2/3 & -2/3 \\ -2/3 & 2/3 & -1/3 & 4/3 \\ 2/3 & 1/3 & -2/3 & 2/3 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$
 (Hint: calculate first the determinant and possible fixed points).

Exercise 11 Consider the rotation in \mathbb{S}^2 given by $f_\alpha : \mathbb{S}^2 \rightarrow \mathbb{S}^2$, $f_\alpha(x, y, z) = (x, y \cos \alpha - z \sin \alpha, y \sin \alpha + z \cos \alpha)$, $x^2 + y^2 + z^2 = 1$.

Show that the function $f_* = \varphi^{-1}f\varphi : \widehat{\mathbb{C}} \rightarrow \widehat{\mathbb{C}}$ (φ is the stereographic projection from the point $X(1, 0, 0)$) induces an isometry T of \mathbb{E}^2 . Determine and describe this isometry.

Exercise 12 Consider the points $P(p_1, p_2)$, $Q(q_1, q_2)$, and $R(r_1, r_2)$ customize by your birthday's parameters. Do a computer animation with translations. First translation T_0 has vector PQ and last translation T_1 has vector PR . Show that any intermediate translation $T(t)$, $0 \leq t \leq 1$, in the animation can be written $T(t) = (1 - t)T_0 + tT_1$. Give the matrix of a general translation in the animation. Use for instance MATLAB, to calculate the translations for $t = j/100$, $0 \leq j \leq 100$

Exercise 13 Show that the rotations $R_1(\mathbf{x}) = q_1xq_1^{-1}$ and $R_2(\mathbf{x}) = q_2xq_2^{-1}$, $x = (0, \mathbf{x})$ given by the quaternions q_1 and q_2 coincide if and only if $q_1 = aq_2$ for some real number $a \neq 0$. Are R_1 and R_2 rotations?

Exercise 14 Determine the latitude λ , $0 < \lambda < \pi/2$, such that the points $A_1(\cos \lambda, 0, \sin \lambda)$, $A_2(0, \cos \lambda, \sin \lambda)$, $A_3(-\cos \lambda, 0, \sin \lambda)$, $A_4(0, -\cos \lambda, \sin \lambda)$, $A_5(\cos \lambda, 0, -\sin \lambda)$, $A_6(0, \cos \lambda, -\sin \lambda)$, $A_7(-\cos \lambda, 0, -\sin \lambda)$, $A_8(0, -\cos \lambda, -\sin \lambda)$ are the vertices of a cube. Determine the great circle through A_2, A_4, A_6, A_8 .