Hand-in Exercises 2, TATA49 Geometry with Applications. Fall 2022

Maple or Matlab may be used in the calculations.

The exercises are customized by your birthdays as in the previous sheet of exercises.

Exercise 1 *Determine the images under the stereographic projection* $\phi : \widehat{\mathbb{C}} \to \mathbb{S}^2$ *of all the circles* $\mathcal{C}_a$ *with equation* $x^2 + y^2 - 2r_2(1+a) + 2ar_1 y = (p_1 + q_2 + 3)^2 - r_2^2(1+a)^2 - r_1^2 a^2$; $a \in \mathbb{R}$ *(a, a parameter).*

Exercise 2 *Determine the images under the inverse of the stereographic projection* $\varphi : \mathbb{S}^2 \to \widehat{\mathbb{C}}$ *of all the great circles* $\mathcal{S}_a$ *passing the points* $E(0,1,0), W(0,-1,0$ *and* $A(\sqrt{1-a^2}, 0, a)$; $1/2 \leq a < 1$.

Exercise 3

1. *Calculate the distances from* $N(0,0,1)$ *to the circle* S_a *in Exercise 2.*
2. *Using spherical coordinates* $(\cos\lambda\cos\theta, \cos\lambda\sin\theta, \sin\lambda)$ *give equations of the circles* S_a *parametrised by the latitude* λ.

Exercise 4 *Determine the planes to which the circles in Exercise 2 belong, and the angles formed, pairwise,*

Exercise 5

1. *Calculate the angles in the spherical triangle with vertices A(longitude* $(r_1-2)r_2$ *deg, latitude 60 deg), B(longitude* $(r_1-2)r_2$ *deg, latitude 30 deg) and C(longitude 0 deg, latitude* $(r_1-2)r_2$ *deg). Observation* $(r_1-2)r_2$ *is a number with two digits:* r_2 *in the units and* $r_1 - 2$ *in the tens. Observe that the calculations are done in radians.*
2. *Calculate the area of the triangle above if we consider that the triangle is on Earth (Radius of Earth is 6371 km).*

Exercise 6

1. *Determine the images under the inverse of the stereographic projection of the great circles in Exercise 5.*
2. *Determine the sizes of the edges of the spherical triangle (on Earth) in exercise 5 (Radius of Earth is 6371 km).*

Exercise 7 *Consider* $\mathbb{R}^4$ *with coordinates* $\mathbf{v} = (v_1, v_2, v_3, v_0)$ *(Observe that the real part of the quaternion is the fourth coordinate). Given the quaternion* $q_0 = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k} + a_0 = ((a_1, a_2, a_3), a_0)$, *show that the multiplication with* q_0 *to the left is a linear transformation in* $\mathbb{R}^4$ *with matrix* $\begin{pmatrix} a_0 & -a_3 & a_2 & a_1 \\ a_3 & a_0 & -a_1 & a_2 \\ -a_2 & a_1 & a_0 & a_3 \\ -a_1 & -a_2 & -a_3 & a_0 \end{pmatrix}$.

Calculate the matrix for the multiplication with q_0 *on the right.*

Exercise 8 1. *Use Exercise 7 to determine the matrix of the rotation in $\mathbb{E}^3$ with quaternion $q_0 = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k} + a_0$, $|q_0| = 1$.*

2. *Use Exercise 7 to determine the matrix of the reflection in $\mathbb{E}^3$ on the plane with equation $\frac{2x_1}{7} + \frac{6x_2}{7} - \frac{3x_3}{7} + 0(1) = 0$. Observe that the last row and column of the matrix equal $(0, 0, 0, 1)$.*

Exercise 9 *Consider an animation of an object with start orientation (time $t = 0$) given by the quaternion $q_s = (p_1 + 3, q_1 + 3, r_1 - 2, r_2 - 1)/l$ where $l = \sqrt{(p_1 + 3)^2 + (q_1 + 3)^2 + (r_1 - 2)^2 + (r_2 - 1)^2}$ and final orientation (time $t = 1$) given by the quaternion $q_f = (7/10, (-1/10, -7/10, 1/10))$. Give the orientations at times $t = j/10$, $j = 0, 1, 2, \ldots, 10$*

Exercise 10 *Identify and describe completely the isometry of $\mathbb{E}^3$ with matrix:*

$$\begin{pmatrix} 1 - \frac{2r_1^2}{r_1^2+r_2^2+(q_1+q_2)^2} & \frac{-2r_1r_2}{r_1^2+r_2^2+(q_1+q_2)^2} & \frac{-2r_1(q_1+q_2)}{r_1^2+r_2^2+(q_1+q_2)^2} & -r_2 \\ \frac{-2r_1r_2}{r_1^2+r_2^2+(q_1+q_2)^2} & 1 - \frac{2r_2^2}{r_1^2+r_2^2+(q_1+q_2)^2} & \frac{-2(q_1+q_2)r_2}{r_1^2+r_2^2+(q_1+q_2)^2} & r_1 + q_1 + q_2 \\ \frac{-2r_1(q_1+q_2)}{r_1^2+r_2^2+(q_1+q_2)^2} & \frac{-2r_2(q_1+q_2)}{r_1^2+r_2^2+(q_1+q_2)^2} & 1 - \frac{2r_3^2}{r_1^2+r_2^2+(q_1+q_2)^2} & -r_2 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

(Hint: Study first the matrix and possible fixed elements).

Exercise 11 *Consider the rotation in $\mathbb{S}^2$ given by $f_\alpha : \mathbb{S}^2 \to \mathbb{S}^2$, $f_\alpha(x, y, z) = (x \cos\alpha + z \sin\alpha, y, -x \sin\alpha + z \cos\alpha)$, $x^2 + y^2 + z^2 = 1$.*

Show that the function $f* = \varphi^{-1} f \varphi : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ (φ is the stereographic projection from the point $E(0, 1, 0)$ on the x_1x_3-plane) induces an isometry T of $\mathbb{E}^2$. Determine and describe this isometry.

Exercise 12 *Consider the points $P(p_1, p_2)$, $Q(q_1, q_2)$, and $R(r_1, r_2)$ customize by your birthday's parameters. Do a computer animation with translations. First translation T_0 has vector PQ and last translation T_1 has vector PR. Show that any intermediate translation $T(t)$, $0 \le t \le 1$, in the animation can be written $T(t) = (1 - t)T_0 + tT_1$. Give the matrix of a general translation in the animation.*

Exercise 13 1. *Let $R_1, \ldots, R_r, S_1, \ldots, S_s$ be reflections in $\mathbb{E}^3$ across planes passing the origin, all the $r + s$ planes contain the origin. Show that if $R_1 \ldots R_r = S_1 \ldots S_s$ then $(-1)^r = (-1)^s$*

2. *Let $x = \mathbf{x}, q = \mathbf{q}$ be non-zero pure quaternions. Show that $qx = xq$ if and only if $\mathbf{x}$ and $\mathbf{q}$ are parallel to each other.*

Exercise 14 *Determine the latitude λ, $0 < \lambda < \pi/2$, such that the points $A_1(\cos\lambda, 0, \sin\lambda), A_2(0, \cos\lambda, \sin\lambda), A_3(-\cos\lambda, 0, \sin\lambda), A_4(0, -\cos\lambda, \sin\lambda)$, $A_5(\cos\lambda, 0, -\sin\lambda), A_6(0, \cos\lambda, -\sin\lambda), A_7(-\cos\lambda, 0, -\sin\lambda), A_8(0, -\cos\lambda, -\sin\lambda)$ are the vertices of a cube. Show that the vertices A_1, A_3, A_6, A_8 are the vertices of a regular tetrahedron. Determine the length of a side of the tetrahedron in $\mathbb{E}^3$ and the spherical distance (in $\mathbb{S}^2$) between A_1 and A_3.*