**Hand-in Exercises 3 i TATA49 Geometry with Applications Fall 2022**

Maple or Matlab may be used in the calculations.

The exercises are customized by your birthdays as follows: $p_1, p_2$ is the month of your birthday, where January is 01, February 02,..., and December 12. $q_1, q_2$ is the date in the month, between 13 and 29, nearest to your birthday. $r_1, r_2$ are the two last digits in the year you were born. However, if you were born in 1990, then $r_1, r_2 = 9, 1$; if you are born in 200i $r_1 = 5 + i, r_2 = 4 + i$. Besides in projective geometry **if you are born October 20, November 22 or December 24 you will be born the 23rd of the corresponding month**.

In the following the coordinates are homogeneous if nothing else is said. Observe that the only distinction in this sheet between points and lines is that points will use (::) and lines [::].

**Exercise 1** *A fixed TV camera records a football match from the (vanishing) point* $V(3(q_1 + q_2), 3(p_1 + p_2), 3r_2, 1)$. *The camera has reference system* $O(4q_1, 4q_2, 4r_2, 1)$, $X = (r_1, 0, -r_1, 0), Y = (3r_1, 4r_1, -3r_1, 0)$. *Calculate the equation of the plane of the camera and give the plane coordinates of the images of the sphere with coordinates* $(3 + r_1 \cos(\phi)\cos(\theta), 3 + r_1 \cos(\phi)\sin(\theta), r_1 \sin(\phi), 1)$. *The plane of the camera is big enough. (All coordinates are affine in* $\mathbb{R}^4$*)*

**Exercise 2** *Consider the pencil* $\mathcal{P}_P$ *of lines through the point* $P(r_1 : r_2 : p_1 + p_2)$.

1. *Give the coordinates* $l[l_1 : l_2 : l_3]$ *of four lines in the pencil.*

2. *Show that the line* $x[1 : 0 : 0]$ *does not belong to the pencil* $\mathcal{P}_P$. *Determine the coordinates of the image of the lines in* $\mathcal{P}_P$ *under the perspectivity* $T_1$ *from* $\mathcal{P}_P$ *to the pencil* $\mathcal{P}_X$ *of lines through* $X(1 : 0 : 0)$ *with axis* $x$.

3. *Show that we can use* $(l_2 : l_3)$ *as homogeneous parameters of the lines in the pencil* $P$ *by having* $y[0 : 1 : 0]$ *and* $z[0 : 0 : 1]$ *as base lines of the pencil* $\mathcal{P}_X$. *Which lines* $a$ *and* $b$ *in the pencil* $\mathcal{P}_P$ *must we use as base lines of the pencil?*

4. *Calculate the cross-ratio of the four lines given in part 1. of the this exercise.*

**Exercise 3** *Consider a second perspectivity* $T_2$ *from the pencil* $\mathcal{P}_X$ *of lines through* $X(1 : 0 : 0)$ *to the pencil* $\mathcal{P}_P$ *in Exercise 2 with axis* $u[1 : 0 : 1]$. *Give the matrix of the perspectivity* $T_2 \circ T_1$ *from the pencil* $\mathcal{P}_P$ *onto itself using as base lines the lines* $a$ *and* $b$ *determined in Exercise 2.3.*

**Exercise 4** *Show that the points* $A(r_2 : p_1 : q_1 + q_2)$, $B(q_2 : r_1 : p_1 + p_2)$, $C(r_2(r_1 + q_2) : (p_1 + r_2)r_1 : (q_1 + q_2)r_1 + (p_1 + p_2)r_2)$ *are collinear. Determine the point* $D$ *collinear with* $A, B, C$ *such that* $R(A, D, C, B) = 2$. *As any line in the real projective plane is a line of real numbers plus its point at infinity, determine the real coordinates of* $A, B, C, D$ *(Of course you have to say first which is your origin and point at infinity).*

**Exercise 5** *An involution* $T$ *is a transformation of order 2, i.e.* $T \cdot T = 1_d$.

1. *Show that a projectivity that is an involution (not the identity) has a matrix* $\begin{pmatrix} a & b \\ c & -a \end{pmatrix}$, *with* $a^2 + bc \neq 0$.

2. *Show that a projectivity on the line* $l$ *that interchanges two points* $P, Q$ *on* $l$ *is an involution. Use an appropriate basis.*

**Exercise 6** *Determine the collineation that sends* $P(p_1 : p_2 : 1)$ *to* $Z(0 : 0 : 1)$, $Q(q_1 : q_2 : 1)$ *to* $X(1 : 0 : 0)$, $R(r_1 : r_2 : 1)$ *to* $Y(0 : 1 : 0)$ *and* $S(p_1 + q_1 + r_1 : p_2 + q_2 + r_2 : 3)$ *to* $T(1, 2, 1)$. *Determine the fixed elements.*

**Exercise 7** *Determine the collineation that sends* $y[0:1:0]$ *to* $r[r_1 : r_2+2 : 0]$, $z[0:0:1]$ *to* $p[p_1 : p_2 : 2]$, $x[1:0:0]$ *to* $q[q_1 : q_2 : 1]$ *and* $t[1:2,1]$ *to* $s[p_1+q_1+r_1 : p_2+q_2+r_2+2 : 3]$. *Determine its fixed elements.*

**Exercise 8** *Consider elations with axis* $m[\sin(\frac{2\pi}{q_1+4}) : \cos(\frac{2\pi}{q_1+4}) : 1]$ *and centre* $C(\cos(\frac{2\pi}{q_1+4}) : -\sin(\frac{2\pi}{q_1+3}) : 0)$. *Show that they are affinities depending on one (1) parameter. Identify these affinities and the geometrical meaning of the parameter, both as collineations and affinities.*

**Exercise 9** *Determine the invariant elements under the collineation with matrix* $\begin{pmatrix} -r_1^2+r_2^2 & 2r_1^2+2r_1r_2 & -2r_1r_2 \\ 0 & a & 0 \\ -2r_1r_2 & 2r_2^2+2r_1r_2 & r_1^2-r_2^2 \end{pmatrix}$. *For which values of* $a$ *is the collineation perspective? In that case is it a homology or an elation? In case the collineation is a homology, determine the cross-ratio of the homology.*

**Exercise 10** *Determine the polarity that sends* $X(1:0:0)$ *to the line* $y[0:1:0]$, *the point* $Z(0:0:1)$ *to the line* $z[0:0:1]$, $R(r_1 : r_2 : 0)$ *to* $r[r_1 : r_2 : 0]$ *and* $S(r_2 : -r_1 : 1)$ *to* $s[3r_2 : -3r_1 : 1])$. *Give the equations of the associated conic of points and conic of tangents. Is any of the points or lines above selfconjugate?*

**Exercise 11** *Let* $T$ *be a collineation. Show that if* $m$ *is the polar line to a point* $P$ *with respect to some conic* $\mathcal{C}$ *with matrix* $C$, *then* $m' = T(m)$ *is the polar to the point* $P' = T(P)$ *with respect to a conic* $\mathcal{C}'$ *with matrix* $C'$. *Control first that* $\mathcal{C}'$ *is a conic.*

**Exercise 12** *Let* $T$ *be a homology whose centre* $Z(0:0:1)$ *and axis* $z[0:0:1]$ *are pole-polar with respect to a polarity with (symmetric) matrix* $C$ *and conic* $\mathcal{C}$. *Calculate the matrix of* $T$. *Show is the homology* $T$ *fixes the polarity and so the conic* $\mathcal{C}$.

**Exercise 13** *Consider the points* $A(1, \sqrt{8})$ *and* $B(-3/5, 4/5)$ *in Poincaré's Upper Half Plane. Determine the geodesic that contains* $A$ *and* $B$.

*Calculate the hyperbolic distance between* $A$ *and* $B$.

**Exercise 14** *Determine the coordinates of the points in a projective plane of order 7. Use triples of the symbols 0, 1, 2, 3, 4, 5 and 6 and arithmetic modulus 7. Determine every line of the plane. How many points does a line have? How many lines and points has the plane?*

*Once you have chosen the line formed by the points* $(x_1 : x_2 : 0)$ *as line at infinity, classify the affine lines according to wether they are parallel or not. How many classes of parallellism are there? How many parallel lines are there in any class of parallelism?*

*Notice that the coordinates* $(1:0:1); (2:0:2); (3:0:3); (4:0:4); (5:0:5); (6:0:6)$ *represent the same point. In the same way* $(1:2:0); (2:4:0); (3:6:0); (4:1:0); (5:3:0); (6:5:0)$ *represent the same point.*