Hand-in Exercises 3 i TATA49 Geometry with Applications Fall 2023

Maple or Matlab may be used in the calculations.

The exercises are customized by your birthdays as follows: p_1, p_2 is the month of your birthday, where January is 01, February 02,..., and December 12. q_1, q_2 is the date in the month, between 13 and 29, nearest to your birthday. r_1, r_2 are the two last digits in the year you were born. However, if you were born in 1990, then $r_1, r_2 = 9, 1$; if you are born in 200i $r_1 = 5 + i, r_2 = 4 + i$. Besides in projective geometry **if you are born October 20, November 22 or December 24 you will be born the 23rd of the corresponding month**.

In the following the coordinates are homogeneous if nothing else is said. Observe that the only distinction in this sheet between points and lines is that points will use (::) and lines [::].

Exercise 1 *Determine the projection of the tetrahedron with vertices* $O(0,0,0,1)$, $A(r_2+p_2,0,0,1), B(0,r_2+p_2,0,1), C(0,r_2+p_2,r_2+p_2,1)$ *om the plane* $x_1+2x_2-2x_3-16(x_4)=0$ *from the (vanishing) point* $V(1/8,1/4,0,1)$. *Calculate the plane coordinates of the images of vertices of the tetrahedron if the coordinate system for the plane is* $O(8r_1,4r_1,0,r_1), X=(2,1,2,0)$ *and* $Y=(2,-2,-1,0)$.

Exercise 2 *Consider the pencil of points with coordinates* $l[r_1+r_2 : r_2+q_2 : r_2]$. *Give a matrix of the projectivity* $T : (l,P,Q) \to (l,P,Q)$ *that consists of the product of the following two perspectivities:*

1. *Perspectivity with centre* $O_1(1:0:r_2)]$ *sending a point in* l *to a point in* $x[1:0:0]$, *followed by*
2. *perspectivity with centre* $O_2(r_1:r_2,1)$ *sending a point in* x *to a point in* l.

Observe that once you choose a basis $\{P,Q\}$ *for* l *you keep the basis to write the matrix of* T.

Exercise 3 *A projective line has affine part the affine line with equation* $(x_1,x_2) = (q_1+3, r_2+5) + t(3(p_1+3), -(p_2+5))$. *Determine the point at infinite of the line and the projective point* $D(d_1:d_2:d_3)$ *on the line such that the cross ratio of the points with parameters* $t=-4,2,-1$ *and* D *is* -1. *What is the parameter for* D*? (use the order of the points as above and notice that the parameter for* D *can be* ∞*).*

Exercise 4 *An involution* T *is a transformation of order 2, i.e.* $T \cdot T = 1_d$.

1. *Show that a projectivity that is an involution (not the identity) has a matrix* $\begin{pmatrix} a & b \\ c & -a \end{pmatrix}$, *with* $a^2+bc \neq 0$.
2. *Show that a projectivity on the line* l *that interchanges two points* P, Q *on* l *is an involution. Use an appropriate basis.*

Exercise 5 *Determine the collineation that sends* $P(p_1-2 : p_2+1 : 1)$ *to* $Y(0:1:0)$*;* $Q(q_1+1 : q_2+1 : 1)$ *to* $Z(0:0:1)$*;* $R(r_1-3 : r_2+1 : 1)$ *to* $X(1:0:0)$ *and finally* $S(p_1+r_1 : q_2+r_2 : 2)$ *till* $U(1:1:1)$. *Determine its fixed elements. Determine and show that the diagonal points of the quandrangle* $PQRS$ *are non-collinear.*

Exercise 6 *Determine the collineation that sends the lines:* $x[1:0:0]$ *to* $q[q_1-2 : q_2+1 : 1]$ *,* $y[0:1:0]$ *to* $p[p_1+3 : p_2+3 : 1]$*;* $z[0:0:1]$ *to* $s[p_1+r_1 : q_2+r_2 : 3]$ *and finally* $u[1:1:1]$ *to* $r[r_1-3 : r_2+1 : 1]$. *Determine the fixed elements. Determine the diagonal lines of the quadrilateral* $qpsr$.

Exercise 7 *Consider elations with axis* $m[r_1 : r_2 : 1]$ *and centre* $C(-r_2 : r_1 : 0)$.. *Show that they are affinities depending on one (1) parameter. Identify these affinities and the geometrical meaning of the parameter, both as collineations and affinities.*

Exercise 8 *Determine the invariant elements under the collineation with matrix* $\begin{pmatrix} -r_1^2 + r_2^2 & 0 & -2r_1r_2 \\ 2r_1^2 + 2r_1r_2 & a & 2r_2^2 + 2r_1r_2 \\ -2r_1r_2 & 0 & r_1^2 - r_2^2 \end{pmatrix}$. *For which values of* a *is the collineation perspective? In that case is it a homology or an elation? In case the collineation is a homology, determine its axis and centre and the cross-ratio of the homology.*

Exercise 9 *Give a polarity such that the triangle* PQR, *with* $P(p_1 + 2 : p_2 : 1)$, $Q(q_2 : q_1 : 0)$, $R(0, : r_2 : r_1)$, *is selfpolar. Give the conic of points and the conic of lines of the polarity. Which kind of conic is it?*

Exercise 10 *Let* T *be a collineation. Show that if* m *is the polar line to a point* P *with respect to some conic* $\mathcal{C}$ *with matrix* C, *then* $m' = T(m)$ *is the polar to the point* $P' = T(P)$ *with respect to a conic* $\mathcal{C}'$ *with matrix* C'. *Control first that* $\mathcal{C}'$ *is a conic.*

Exercise 11 *Let* T *be a homology whose centre* $Z(0 : 0 : 1)$ *and axis* $z[0 : 0 : 1]$ *are pole-polar with respect to a polarity with (symmetric) matrix* C *and conic* $\mathcal{C}$. *Calculate the matrix of* T. *Show is the homology* T *fixes the polarity and so the conic* $\mathcal{C}$. *(Hint, there are infinitely many such homologies)*

Exercise 12 *Consider the points* $A(-12/13, 5/13)$ *and* $B(3/5, 4/5)$ *in Poincaré's Upper Half Plane.*

1. *Determine the geodesic that contains* A *and* B.
2. *Calculate the hyperbolic distance between* A *and* B.

Exercise 13 *Consider the hyperbolic triangle with vertices* $A(-12/13, 5/13)$, $B(3/5, 4/5)$ *and* $C(3/5, 9/5)$ *in Poincaré's Upper Half Plane.*

1. *Give the angles of the triangle at the vertices. (Hint, remember that the slope to the tangent to the circle with equation* $(x - c)^2 + y^2 = r^2$ *at the point* (x_0, y_0) *is given by the equation* $(x - x_0) + y_0y' = 0$.
2. *Calculate the area of the triangle.*

Exercise 14 *Determine the coordinates of the points in a projective plane of order 4. Use triples* (x_1, x_2, x_3) *of the symbols 0, 1,* α *and* $\alpha + 1$ *and the equivalence relation* $(x_1, x_2, x_3) \sim \lambda(x_1, x_2, x_3)$ *with* $\lambda = 1, \alpha,$ *or* $\alpha + 1$. *Recall that* $(\alpha)(\alpha + 1) = 1$, $(\alpha)(\alpha) = \alpha + 1$, $(\alpha + 1)(\alpha + 1) = \alpha$ *Determine every line of the plane. How many points does a line have? How many lines and points has the plane?*

Once you have chosen the line formed by the points $(x_1 : x_2 : 0)$ *as line at infinity, classify the affine lines according to wether they are parallel or not. How many classes of parallellism are there? How many parallel lines are there in any class of parallelism?*

Notice that the coordinates $(1 : 0 : 1); (\alpha : 0 : \alpha); (\alpha + 1 : 0 : \alpha + 1)$ *represent the same point. In the same way* $(1 : \alpha : \alpha + 1); (\alpha : \alpha + 1 : 1); (\alpha + 1 : 1 : \alpha)$ *represent the same point.*