Exercises on Quaternions, 3D and Polyhedra

Exercise 1 *Consider an animation for change of orientation of an object, i.e. rotations. Start orientation* R_0 *gives of the unit quaternion* $q_0 = (2/7, 3/7, 0, 6/7)$ *and last orientation* R_1 *has quaternion* $q_1 = (1/2, 1/2, -1/2, 1/2)$. *Give a general quaternion* $R(t)$, $0 \leq t \leq 1$, *in the animation using SLERP. Use MATLAB to calculate the quaternions for* $t = j/100, 0 \leq j \leq 100$.

Exercise 2 *Determine the matrix and the quaternion of the reflection on the plane* $6x_1 - 3x_2 - 2x_3 = 0$.

Exercise 3 *Consider the quaternions* $q_1 = (0, 1/\sqrt{3}, 1/\sqrt{3}, 1/\sqrt{3})$ *and* $q_2 = (0, 5/12, 7/12, 0)$. *Let* $S_{q_i}(p) = -q_i p \overline{q}_i$ *be reflections. Determine the axes and angles of the rotations* S_2S_1 *and* S_1S_2. *Numerical approximation for there angles.*

Exercise 4 *Consider rotations with axis through origin given by the quaternions* $q_1 = (3/7, 0, 6/7, 2/7)$ *and* $q_2 = (4/9, 1/9, 8/9, 0)$. *Determine the rotations with quaternions* q_1q_2 *och* q_2q_1.

Exercise 5 *Consider the regular polygons* P_n *and* P_{2n} *with* n *and* $2n$ *sides respectively. Show that the symmetry group of* P_n *is a subgroup of the symmetry group of* P_{2n}. *Which transformations are in the second symmetry group and not in the first one?*

Exercise 6 *Show that the affine transformation of* $\mathbb{E}^3$ *with matrix*

$$\begin{pmatrix} \frac{2\sqrt{2}+3}{6} & \frac{2\sqrt{2}-3}{6} & \frac{1}{3\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{2\sqrt{2}-3}{6} & \frac{2\sqrt{2}+3}{6} & \frac{1}{3\sqrt{2}} & 0 \\ \frac{1}{3\sqrt{2}} & \frac{1}{3\sqrt{2}} & \frac{-4}{3\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

is an isometry of $\mathbb{E}^3$. *Decompose it in a product of reflections. How many reflections?*

Exercise 7 *Identify the isometry with matrix:*

$$\begin{pmatrix} -1/3 & (1-\sqrt{3})/3 & (-1-\sqrt{3})/3 & -1/3 \\ (1+\sqrt{3})/3 & -1/3 & (1-\sqrt{3})/3 & -2/3 \\ (-1+\sqrt{3})/3 & (1+\sqrt{3})/3 & -1/3 & 2/3 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Does the isometry fix some plane? Which?

Exercise 8 *Determine if the affine transformation of* $\mathbb{E}^3$ *with matrix*

$$\begin{pmatrix} 6/7 & 2/7 & 3/7 & 1/14 \\ 2/7 & 3/7 & -6/7 & -1/7 \\ -3/7 & 6/7 & 2/7 & 1/7 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

is a screw-motion or a rotation of $\mathbb{E}^3$. *Decompose it in a product of reflections. Determine the axis, and the angle of the motion.*

Exercise 9 *Determine the matrix of the affine transformation of* $\mathbb{E}^3$ *that takes* $P(0,0,1)$ *to* $P'(1,0,0)$*,* $Q(-1,0,-1)$ *to* $Q'(1,1,0)$*,* $R(1,2,3)$ *to* $R'(1,1,1)$*, and* $S(1,0,0)$ *to* $S'(0,1,0)$.

Exercise 10 *Consider the unit quaternion* $q = (\cos\theta, \sin\theta\mathbf{v})$*, with* $|\mathbf{v}| = 1$*. Given a puer quaternion* $p = (0, \mathbf{p})$ *we define* $C_q(\mathbf{p}) = qpq^{-1}$.

Show that $C_q(\mathbf{p}) = (0, \cos 2\theta\mathbf{p} + 2(\sin\theta)^2(\mathbf{p}\cdot\mathbf{v})\mathbf{v} - \sin 2\theta(\mathbf{p}\times\mathbf{v}))$.

Exercise 11 *Let* $p = (0, \mathbf{p})$ *and* $q = (0, \mathbf{q})$ *be puer quaternions such that* $\mathbf{p}$ *orthogonal to* $\mathbf{q}$*. Show that* $-qpq^{-1} = p$

Exercise 12 *Show that the tetrahedron with vertices* $A(2,2,2,1), B(2,-2,-2,1), C(-2,-2,2,1), D(-2,2,-2,1)$ *is regular.*

Give the matrix of the antipodal transformation $\dashv$*, also called reflection across the origin* $O(0,0,0,1$*, defined by* $\dashv(P(x,y,z)) = P'(-x,-y,-z)$.

Show that $\dashv$ *is a symmetry of the tetrahedron above.*