1) Calculate the hyperbolic distance between $z = \frac{1}{2} + \frac{i\sqrt{3}}{2}$ and $w = -\frac{1}{2} + \frac{i\sqrt{3}}{2}$

Method 1: Using $\sinh^2\left(\frac{1}{2} d_H\right) = \frac{|z-w|^2}{4 \operatorname{Im} z \operatorname{Im} w}$

we have $\sinh^2\left(\frac{1}{2} d\right) = \frac{\left(\frac{1}{2} - \left(-\frac{1}{2}\right)\right)^2 + 0^2}{4 \frac{\sqrt{3}}{2} \frac{\sqrt{3}}{2}} = \frac{1}{3}$

$$\sinh\left(\frac{1}{2} d\right) = \frac{1}{\sqrt{3}} \; ; = \frac{\left(e^{\frac{1}{2}d}\right)^2 - 1}{2 e^{\frac{1}{2}d}} \qquad \left(\text{change of var } e^{\frac{1}{2}d} = y\right)$$

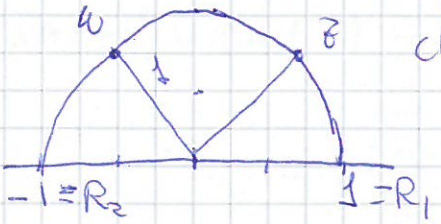
$$\frac{2}{\sqrt{3}} y = y^2 - 1 \; ; \; y^2 - \frac{2}{\sqrt{3}} y - 1 = 0$$

$$y = \frac{\frac{2}{\sqrt{3}} \pm \sqrt{\frac{4}{3} + 4}}{2} = \frac{1}{\sqrt{3}} \pm \frac{2}{\sqrt{3}} \Rightarrow y = \sqrt{3}$$

$$e^{\frac{1}{2}d} = 1/\frac{1}{\sqrt{3}} \; ; \; \frac{1}{2} d = \left|\ln \frac{1}{\sqrt{3}}\right| \; ; \; d = 2 \ln \sqrt{3}$$

$$d = \ln 3$$

Method 2: Using the cross-ratio, z and w belong to the Euclidean unit circumference



$$d_H(z,w) = |\ln R(z, w, R_2, R_1)| =$$

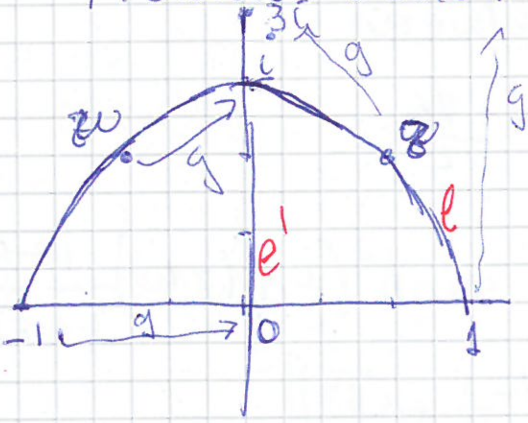
$$= \left|\ln R\left(\frac{1}{2} + \frac{\sqrt{3} i}{2}, -\frac{1}{2} + \frac{i\sqrt{3}}{2}, -1, 1\right)\right|$$

$$\cancel{= \left|\ln \frac{\frac{1}{2} + \frac{i\sqrt{3}}{2}}{\frac{1}{2} - \frac{i\sqrt{3}}{2}}\right| = \left|\ln\left(\frac{(1+\sqrt{3})(1+\ldots)}{}\right.\right.}$$

$$= \left|\ln \frac{\left(-\frac{3}{2} - \frac{i\sqrt{3}}{2}\right)\left(\frac{3}{2} - \frac{i\sqrt{3}}{2}\right)}{\left(-\frac{1}{2} - \frac{i\sqrt{3}}{2}\right)\left(\frac{1}{2} - \frac{i\sqrt{3}}{2}\right)}\right| = \left|\ln \frac{-3(\sqrt{3}+i)(\sqrt{3}-i)}{(1+i\sqrt{3})(1-i\sqrt{3})}\right|$$

$$= \left|\ln\left(-3 \frac{4}{-4}\right)\right| = |\ln 3| = \ln 3$$

Method 3 Mapping the hyperbolic line containing z and w to the imaginary axis with a Mobius transformation



$$g(z) = \frac{az+b}{cz+d} \qquad \begin{cases} g(-1)=0 \Rightarrow -a+b=0 \\ g(1)=\infty \Rightarrow c+d=0 \\ ad-bc=1 \;;\; 2ad=1 \\ b=a,\ d=\frac{1}{2a},\ c=\frac{-1}{2a} \end{cases}$$

$$g\left(-\frac{1}{2}+\frac{i\sqrt{3}}{2}\right) = i$$

$$\Leftrightarrow \frac{a\left(\frac{-1}{2}+\frac{i\sqrt{3}}{2}+1\right)}{\frac{1}{2a}\left(-\left(-\frac{1}{2}+\frac{i\sqrt{3}}{2}\right)+1\right)} = i$$

$$i = \frac{2a^2\,\frac{1}{2}(1+i\sqrt{3})}{\frac{1}{2}(3-i\sqrt{3})} \;;$$

$$a^2 = \frac{\sqrt{3}}{2}\, i\, \frac{(\sqrt{3}-i)}{(1+i\sqrt{3})} = \frac{\sqrt{3}}{2}\, i\, \frac{(\sqrt{3}-i)(1-i\sqrt{3})}{4}$$

$$= \frac{\sqrt{3}}{2} \qquad ; \quad a = \sqrt{\frac{\sqrt{3}}{2}}$$

$$g(z) = 2a^2\left(\frac{z+1}{-z+1}\right) = \sqrt{3}\left(\frac{z+1}{-z+1}\right)$$

$$g\left(\frac{1}{2}+\frac{i\sqrt{3}}{2}\right) = \sqrt{3}\left(\frac{\frac{3}{2}+\frac{i\sqrt{3}}{2}}{\frac{1}{2}-\frac{i\sqrt{3}}{2}}\right) = \frac{3(\sqrt{3}+i)(1+i\sqrt{3})}{4} = 3i$$

$$d_H(z,w) = d_H(w,z) = \left|\ln R(i,3i,0,\infty)\right| = \left|\ln\left(\frac{-i}{-i3}\right)\right| = |-\ln 3|$$

$$= \ln 3$$

Obs: We can use the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, but then we need bases for the projective lines l (containing z and w) and l' (imaginary axis)

2) Calculate the hyperbolic distance between

$z = \frac{1+i}{8}$, and $w = \frac{7+i}{8}$

<u>Method 1</u> $\sinh^2\left(\frac{1}{2} d_H\right) = \frac{\left|-6/8\right|^2}{4 \cdot \frac{1}{8} \cdot \frac{1}{8}} = 9$,

$$\sinh^{\cancel{2}}\left(\tfrac{1}{2} d\right) = 3$$

$$\sinh\left(\tfrac{1}{2} d\right) = \frac{\left(e^{\frac{1}{2}d}\right)^2 - 1}{2e^{\frac{1}{2}d}} = 3, \quad \left(y = e^{\frac{1}{2}d}\right)$$

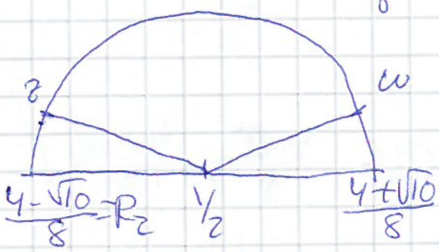
$$6y = y^2 - 1 \; ; \quad y^2 - 6y - 1 = 0$$

$$y = \frac{6 \pm \sqrt{36+4}}{2} = 3 \pm \sqrt{10}, \quad y = 3 + \sqrt{10}$$

$$\tfrac{1}{2} d = \ln(3+\sqrt{10}), \quad d = 2\ln(3+\sqrt{10})$$

$$d = \ln(3+\sqrt{10})^2 = \ln(19 + 6\sqrt{10})$$

<u>Method 2</u> the hyperbolic line through z and w is the half circumference with centre $z_c = \frac{1}{2}$ and radius $r = \frac{\sqrt{10}}{8}$



$$d_H(z,w) = \cancel{\pm}\left|\ln R\left(\frac{7+i}{8}, \frac{1+i}{8}, \frac{4-\sqrt{10}}{8}, \frac{4+\sqrt{10}}{8}\right)\right|$$

$$= \left|\ln \frac{(-3-\sqrt{10}-i)(3+\sqrt{10}-i)}{(3-\sqrt{10}-i)(-3+\sqrt{10}-i)}\right| =$$

$$= \left|\ln \frac{(3+\sqrt{10}+i)(3+\sqrt{10}-i)}{(-3+\sqrt{10}+i)(-3+\sqrt{10}-i)}\right| =$$

$$= \left|\ln \frac{(3+\sqrt{10})^2+1}{(-3+\sqrt{10})^2+1}\right| = \left|\ln \frac{1+9+10+6\sqrt{10}}{1+9+10-6\sqrt{10}}\right| = \left|\ln \frac{\sqrt{10}+3}{\sqrt{10}-3}\right|$$

$$= \left|\ln -\frac{(\sqrt{10}+3)^2}{9-10}\right| = \left|\ln(\sqrt{10}+3)^2\right| = 2\ln(\sqrt{10}+3)$$

$$= \ln(19 + 6\sqrt{10})$$