Isometries of $\mathbb{E}^3$

Def. An isometry of $\mathbb{E}^3$ (seen as the affine space $x_4 = 1$ in $\mathbb{R}^4$) is an affine transformation that preserves distances

Th. Given an isometry $T: \mathbb{E}^3 \to \mathbb{E}^3$ the map $\bar{T}(P) = T(P) - T(O)$ is a linear map

i) $\bar{T}(\bar{0}) = \bar{0}$

ii) $\bar{T}$ is an affinity since it is the product of T with the translation of vector $-T(0)$. $|\bar{T}(x)| = |T(x) - T(0)|$
$= |x - 0| = |x|$

~~Tau iso~~ T isometry = product of a linear isometry with a translation.

Matrix representation $A_T = \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ 0 & 0 & 0 & 1 \end{pmatrix}$ with $A_{\bar{T}} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$

orthogonal ($A_{\bar{T}}^{-1} = A_{\bar{T}}^{t}$)

$\det A_T (= \det A_{\bar{T}}) = 1$ direct

$\det A_T (= \det A_{\bar{T}}) = -1$ indirect

- The images of four pts, not co-planar, pts determine the affinity (and so the isometry)

Th. An isometry of $\mathbb{E}^3$ is the product of ~~largest~~ at most four reflections. If it fixes O, at most three reflections.

Examples translations $T_a(P) = P + a$ $\quad a = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$

$A_T = \begin{pmatrix} 1 & 0 & 0 & a_1 \\ 0 & 1 & 0 & a_2 \\ 0 & 0 & 1 & a_3 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

Reflection across a plane π: fixes the pts of π and given $P \notin \pi$ π is the bisector of the segment PP' ($P' = R(P)$)

π given by $x \cdot n = d$ $\quad (|n| = 1)$

$$R(x) = x + 2(d - x \cdot n)\, n \quad \text{(the reflection in } \pi\text{)}$$

Exercise Prove it $A = \begin{pmatrix} 1-2n_1^2 & -2n_2n_1 & -2n_3n_1 & 2dn_1 \\ -2n_1n_2 & 1-n_2^2 & -2n_3n_2 & 2dn_2 \\ -2n_1n_3 & -2n_2n_3 & 1-n_3^2 & 2dn_3 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

Translations can be expressed as the product of two reflections across parallel planes

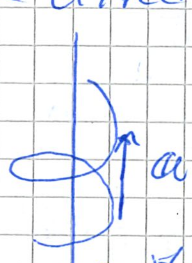
$\mathbb{x} \cdot \mathbb{n} = d_2$

$\mathbb{x} \cdot \mathbb{n} = d_1$

$\mathbb{a} = 2(d_1 - d_2)\mathbb{n}$

The product of two reflections R_1, R_2 in two intersecting planes Π_1, Π_2 is a rotation with axis $l = \Pi_1 \cap \Pi_2$ and angle the double of the dihedral angle between Π_1 and Π_2

- A screw-motion in $\mathbb{E}^3$ is a rotation about some axis followed by a translation in the direction of the axis (helices are invariant curves)



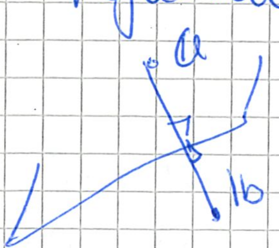
Exercise If an isometry T fixes $0, \mathbb{x}_1\begin{pmatrix}1\\0\\0\end{pmatrix}, \mathbb{x}_2\begin{pmatrix}0\\1\\0\end{pmatrix}, \mathbb{x}_3\begin{pmatrix}0\\0\\1\end{pmatrix}$

then $T = Id$

($T(\mathbb{x}) = \mathbb{y}$ $\quad \|\mathbb{y} - \mathbb{x}_1\| = \|T\mathbb{x} - T\mathbb{x}_1\| = |\mathbb{x} - \mathbb{x}_1|$ and

$\mathbb{y} \cdot \mathbb{x}_1 = \mathbb{x} \cdot \mathbb{x}_1$. Similarly for $\mathbb{y} \cdot \mathbb{x}_2 = \mathbb{x} \cdot \mathbb{x}_2$, $\mathbb{y} \cdot \mathbb{x}_3 = \mathbb{x} \cdot \mathbb{x}_3$

Exercise Suppose that $|\mathbb{a}| = |\mathbb{b}| \neq 0$. Then there is a reflection across a plane Π s.t. $R(\mathbb{a}) = \mathbb{b}$, $R(\mathbb{b}) = \mathbb{a}$



$\mathbb{n} = \frac{\mathbb{a} - \mathbb{b}}{|\mathbb{a} - \mathbb{b}|}$

$$A_{R_\Pi} = \left(\begin{array}{c|c} \mathbb{x} - (2\mathbb{x} \cdot \mathbb{n})\mathbb{n} & 2d\mathbb{n} \\ 0 & 1 \end{array}\right)$$

Th Any direct isometry is a screw-motion

a) We need to consider $T(\mathbb{x}) = \mathbb{x}_0 + L(\mathbb{x})$, L a rotation. Consider the axis of L and a vector $\mathbb{a}$ lying on the axis if $x_0 = t\,a$, done

$x_0 \neq t\,a$ for any scalar t

Any rotation with axis parallel to the axis of L and the same angle

$\tilde{L}(x) = y_0 + L(x - y_0)$

So $L(x) = y_0 + L(x - y_0) + \lambda a$ for some y_0 and λ

$x_0 + L(x) = y_0 + L(x) - L(y_0) + \lambda a$

$x_0 = y_0 - L(y_0) + \lambda a \qquad x_0 = x_1 + t\,a$

$x_1 \perp a$

if $y_0 - L(y_0) + \lambda a = x_1 + t\,a$

so $\lambda = t$ and $x_1 = y_0 - L(y_0)$ (always exists in the plane $x \cdot a = c$)

The product of two rotations is a rotation

Which axis, which angle?

Let Π_0 the plane containing the axes of R_1 and R_2

Let α_0 be the reflection across Π_0

Then $R_1 = \alpha_0 \alpha_1$ with α_1 another reflection (such that good angle), similarly $R_2 = \alpha_2 \alpha_0$

$R_2 R_1 = \alpha_2 \alpha_1$

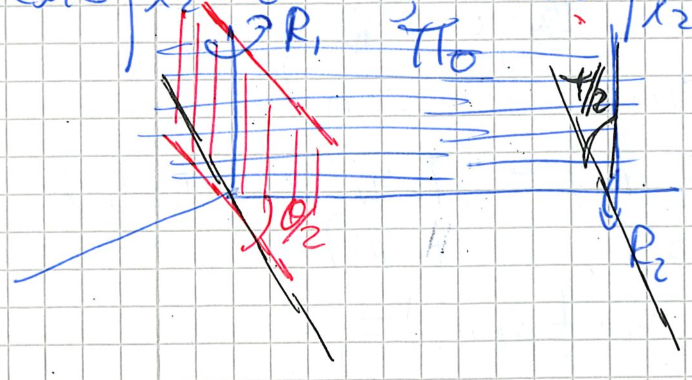
α_1 reflection across plane $\Pi_1 \quad \measuredangle(\Pi_1, \Pi_0) = \frac{1}{2}\theta_{R_1}$

α_2 reflection across plane $\Pi_2 \quad \measuredangle(\Pi_0, \Pi_2) = \frac{1}{2}\theta_{R_2}$

Axis of $R_2 R_1 = \Pi_1 \cap \Pi_2$, $\quad \theta_{R_1 R_2} = 2\measuredangle(\Pi_1, \Pi_2)$

<u>Example</u> $A_{R_1} = \begin{pmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ & & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad A_{R_2} = \begin{pmatrix} -\sin\psi & 0 & \cos\psi & 0 \\ 0 & 1 & 0 & 0 \\ \cos\psi & 0 & \sin\psi & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

axis $\begin{cases} x_1 = 0 \\ x_2 = 0 \end{cases}$ θ $\qquad$ axis $\begin{cases} x_1 = 0 \\ x_2 = 3 \end{cases}$ $\qquad \Pi_0 \to x_1 = 0$



$-\ R_n(x) = -n x n^{-1}$ Reflection across the plane $x \cdot n = 0$

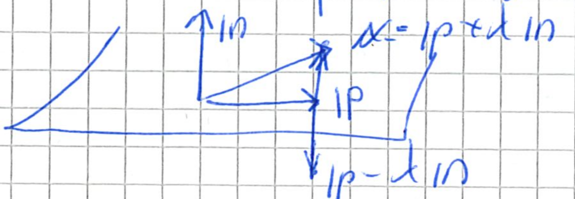
$n = (0, n)$, $x = (0, x)$

Invariant pts $S_n(x) = -n x n^{-1} = x$ or $-nx = xn$

true iff $x \perp n$ $\quad (0, -(n \cdot x) n + n \times (x \times n))$

$x = p + \lambda n$, $p \perp n$ $\quad S_n(x) = -n(p + \lambda x) n^{-1}$

$= -n p n^{-1} - \lambda n = p - \lambda n$



Example Reflection in the $x_1 x_3$-plane $n = j$

$S(x) = -j x j^{-1} = +(0, (0,1,0))(0, (x_1, x_2, x_3))(0, (0,1,0))$

$= (-x_2, (x_3, 0, -x_1))(0, (0,1,0)) = (0, (+x_1, -x_2, x_3))$

- Reflection in the plane ~~with normal~~ $x_1 + 2x_2 + 2x_3 = 0$

Rotations

$R = S_{q_2} S_{q_1}$ with $|q_1| = |q_2| = 1$ $\quad q_i = (0, q_i)$

$R(x) = -q_2(-q_1 x q_1^{-1}) q_2^{-1} = (q_2 q_1) x (q_2 q_1)^{-1}$

as $|q_2 q_1| = 1 \to (q_2 q_1)^{-1} = \overline{q_2 q_1} = \bar{q}_1 \bar{q}_2$

The rotation with axis n and angle θ is given by $q = (\cos \frac{1}{2}\theta, \sin \frac{1}{2}\theta\, n)$, $|n| = 1$

Example Rotation $R_j = -j x j = j x (-j) = j x j^{-1}$

$j = (\cos \alpha/2, \sin \alpha/2\, (0,1,0))$ $\quad R_j$ is a rotation with axis x_2-axis and angle α

The product of two rotations is a rotation

$R_{q_2} R_{q_1}(x) = (q_2 q_1) x (\bar{q}_1 \bar{q}_2) = (q_2 q_1) x (\overline{q_2 q_1})$

rotation $q = q_2 q_1 = (\cos t, \sin t\, n)$

$(\cos \frac{1}{2}\theta_2, \sin \frac{1}{2}\theta_2\, n_2)(\cos \frac{1}{2}\theta_1, \sin \frac{1}{2}\theta_1\, n_1) =$