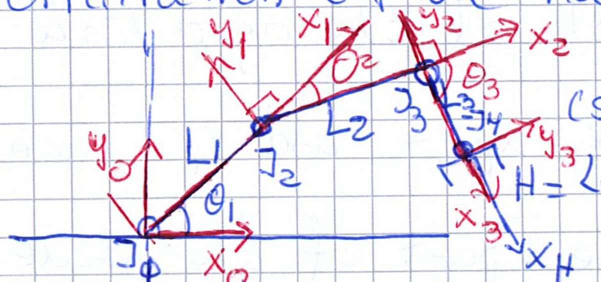
Plane Robotics

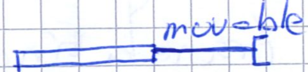
Space of possible configurations of mechanical linkages, ~~e.g.~~ robot arms

Aim: To describe and specify the position and orientation of the hand (effector)



Robot consists of rigid links (segments) connected by joints

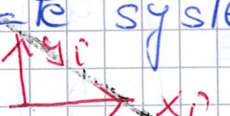
The position and orientation of the hand will depend of the rotational geometry of the junctions (and the length of the segments). We do not consider prismatic junctions



Denavit - Hartenberg notation describes how each rigid link is related to the previous link by mean of isometries of $\mathbb{E}^2$

Space of configurations of the arm is parametrized by the angles θ_i between links with local coordinates system (x_i, y_i) makes with the next rigid link.

If we take all the local coordinate systems to be ON (and right-hand oriented



) the angle θ_i is $\measuredangle(L_{i-1}, L_i)$ oriented counterclockwise

$$J = \underbrace{S^1 \times \cdots \times S^1}_{\text{number of joints}}$$

{ Notice that we may have constrains and we will have subsets of S^1

If there are prismatic junction of length in intervals $I_1, \ldots I_p$

$$J = \underbrace{S^1 \times \cdots \times S^1}_{\text{no. revolute junctions}} \times I_1 \times \cdots \times I_p$$

At a junction $J_i(L_{i-1}, L_i)$, we have

(x_i, y_i) $\begin{cases} * \ x_i \text{ - direction of link after junction } i \\ * \ y_i \text{ - orthogonal direction (right-hand side)} \end{cases}$

* $\forall i \geq 1$ (x_{i-1}, y_{i-1}) coordinates of junction i $(l_{i-1}, 0)$ where l_{i-1} is the length of the rigid link L_{i-1}

* θ_i is the angle of the x_{i-1}- and x_i- axes

Now, a pt P on the plane (part of the plane) with coordinates (a_i, b_i) in the ref. system (J_i, x_i, y_i) has coordinates (a_{i-1}, b_{i-1}) in the ref system $(J_{i-1}, x_{i-1}, y_{i-1})$ that are given by <u>first rotating</u> θ_i and then translating $(l_{i-1}, 0)$

$$\begin{pmatrix} a_{i-1} \\ b_{i-1} \\ 1 \end{pmatrix} = \underbrace{\begin{pmatrix} \cos\theta_i & -\sin\theta_i & l_{i-1} \\ \sin\theta_i & \cos\theta_i & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{A_i} \begin{pmatrix} a_i \\ b_i \\ 1 \end{pmatrix}$$

Then we can control (using the reference system (J_1, x_0, y_0)) the coordinates as seen by the hand (with local coordinates $(x_h, y_h) = (x_n, y_n)$)

$$f : J \longrightarrow \mathcal{C} = \mathcal{U} \times V \qquad \mathcal{U} \subseteq \text{plane}$$

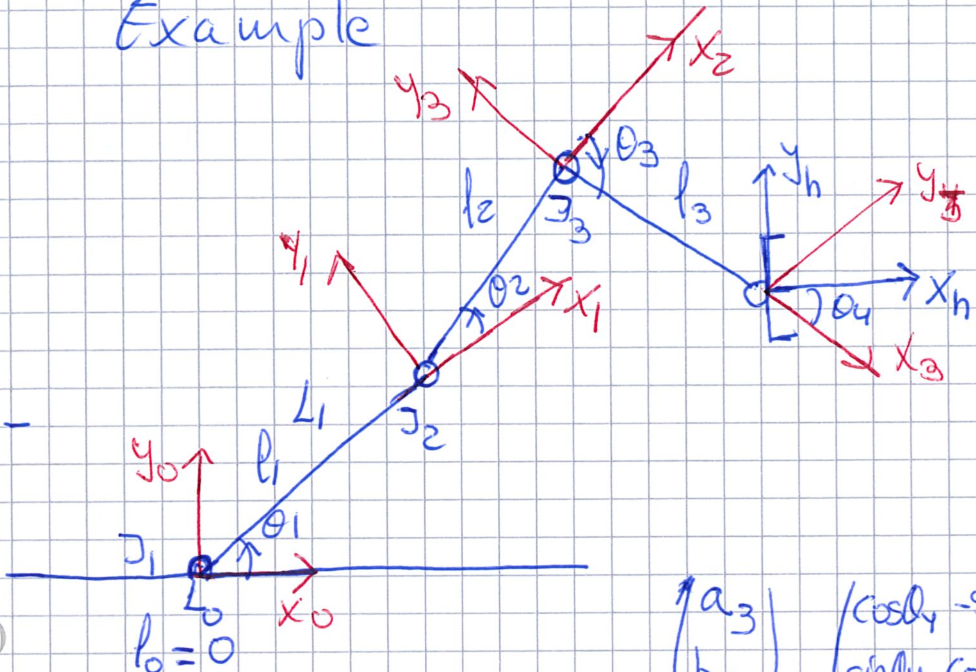
$$(\theta_1, \ldots, \theta_n) \longmapsto (a, b, w), \quad |w| = 1 \ (\text{total angle})$$

Where $\begin{pmatrix} a \\ b \\ 1 \end{pmatrix} = A_1 A_2 \cdots A_{n-1} A_n \begin{pmatrix} a_h \\ b_h \\ 1 \end{pmatrix}$

and $w = (\cos\theta, \sin\theta)$, where $\theta = \sum_{i=1}^{n} \theta_i$

Notice that (a, b) are functions of $\theta_1, \ldots, \theta_n$ since $A_i(\theta_i)$

Example



$$\begin{pmatrix} a_3 \\ b_3 \\ 1 \end{pmatrix} = \begin{pmatrix} \cos\theta_4 & -\sin\theta_4 & l_3 \\ \sin\theta_4 & \cos\theta_4 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a_h \\ b_h \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} a_2 \\ b_2 \\ 1 \end{pmatrix} = \begin{pmatrix} \cos\theta_3 & -\sin\theta_3 & l_2 \\ \sin\theta_3 & \cos\theta_3 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a_3 \\ b_3 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} a_1 \\ b_1 \\ 1 \end{pmatrix} = \begin{pmatrix} \cos\theta_2 & -\sin\theta_2 & l_1 \\ \sin\theta_2 & \cos\theta_2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a_2 \\ b_2 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} a \\ b \\ 1 \end{pmatrix} = \begin{pmatrix} \cos\theta_1 & -\sin\theta_1 & 0 \\ \sin\theta_1 & \cos\theta_1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} a \\ b \\ 1 \end{pmatrix} = A_1 A_2 A_3 A_4 \begin{pmatrix} a_h \\ b_h \\ 1 \end{pmatrix}$$

In applications: Given a point (a,b) and an orientation θ. Is it possible to place the hand of the robot at that place with that orientation??

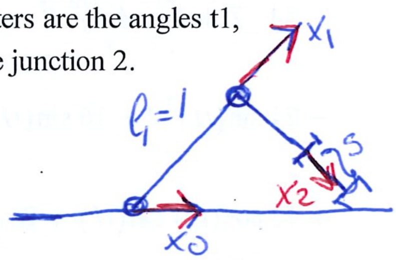
With other words given $c = (a, b, \theta) \in \mathcal{C}$, what is $f^{-1}(c)$??

We need to find $\theta_1, \ldots, \theta_n$ s.t. $\theta = \sum_{i=1}^{n} \theta_i$, $\begin{pmatrix} a \\ b \end{pmatrix} = f(\theta_1, \ldots, \theta_n)$

$with(plots):$
$with(LinearAlgebra):with(SolveTools);$

[AbstractRootOfSolution, Basis, CancelInverses, Combine, Complexity, Engine, GreaterComplexity, Identity, Inequality, Linear, Parametric, Polynomial, PolynomialSystem, RationalCoefficients, SemiAlgebraic, SortByComplexity] **(1)**

5) This example provides the matirx of the configuration of a planar robot with two junctions (angles t1, t2), a rigid link of length l and periscopic hand of length s, with s∈ [0, s1]. The matrix is the product, in this order, A_{h2} the translation matrix with vector (s,0), A_{21} is the matrix of the rotation taking the system from link 1 to the hand, junction 2, followed by the translation with length the length of link 1, finally the matrix A_{10} is the matrix of the rotation with angle the angle at junction 1. The parameters are the angles t1, t2, the length l of the link and the parameter s of the distance between the hand and the junction 2.



$Ah2 := Matrix([[1, 0, s], [0, 1, 0], [0, 0, 1]]):$
$A21 := Matrix([[\cos(t2), -\sin(t2), l], [\sin(t2), \cos(t2), 0], [0, 0, 1]]):$
$A10 := Matrix([[\cos(t1), -\sin(t1), 0], [\sin(t1), \cos(t1), 0], [0, 0, 1]]):$
$Conf := MatrixMatrixMultiply(A10, MatrixMatrixMultiply(A21, Ah2));$

$$Conf := [[\cos(t1)\cos(t2) - \sin(t1)\sin(t2), -\cos(t1)\sin(t2) - \sin(t1)\cos(t2), \cos(t1)(\cos(t2)s + l) - \sin(t1)\sin(t2)s], [\sin(t1)\cos(t2) + \cos(t1)\sin(t2), \cos(t1)\cos(t2) - \sin(t1)\sin(t2), \sin(t1)(\cos(t2)s + l) + \cos(t1)\sin(t2)s], [0, 0, 1]] \quad (2)$$

Observe that the configuration gives the total angle t1+t2 and the vector providing the coordinates of the hand. It gives the relation between the coordinates of a point as seen by the hand, Xh, and the coordinates of the same point as seen by the controller, X0: ConfXh =X0

Using the command **solve** we can calculate (numerically) the parameters t1, t2 and s (l is a constant of the system) for given Xh and X0. This is what the controller does to move the hand properly, giving the angles and the lenght of the periscope. This problem is much more complicated

$$solve\left(\left\{\frac{\cos(t1+t2)\cdot 1}{2} - \frac{\sin(t1+t2)\cdot 1}{2} + \cos(t1)\cdot(\cos(t2)\cdot s + 2) - \sin(t1)\cdot\sin(t2)\cdot s = 3, \frac{\sin(t1+t2)\cdot 1}{2} + \frac{\cos(t1+t2)\cdot 1}{2} + \sin(t1)\cdot(\cos(t2)\cdot s + l) + \cos(t1)\cdot\sin(t2)\cdot s = 3\right\}, [t1, t2, s]\right);$$

$$\Big[\Big[t1 = t1, t2 = \arctan\Big(RootOf\big((4\sin(t1)^2 l^2 - 24\sin(t1)\, l - 16\sin(t1)^2 - 48\cos(t1) + 88)\_Z^2 + (-4\sin(t1)^2 l + 8\sin(t1)^2 + 12\cos(t1) + 12\sin(t1) - 8)\_Z + 4\sin(t1)^4 l^2 - 16\sin(t1)^4 l - 24\cos(t1)\, l\sin(t1)^2 - 4\sin(t1)^2 l^2 - 24\sin(t1)^3 l \quad (3)$$

$$+ 16 \sin(t1)^4 + 48 \cos(t1) \sin(t1)^2 + 16 \sin(t1)^2 l + 48 \sin(t1)^3 + 72 \cos(t1) \sin(t1)$$

$$+ 24 \sin(t1) l - 16 \sin(t1)^2 - 48 \sin(t1) - 35), ((2 \sin(t1)^2 l - 4 \sin(t1)^2$$

$$- 6 \sin(t1) - 6 \cos(t1) + 4) \mathit{RootOf}((4 \sin(t1)^2 l^2 - 24 \sin(t1) l - 16 \sin(t1)^2$$

$$- 48 \cos(t1) + 88) \_Z^2 + (-4 \sin(t1)^2 l + 8 \sin(t1)^2 + 12 \cos(t1) + 12 \sin(t1)$$

$$- 8) \_Z + 4 \sin(t1)^4 l^2 - 16 \sin(t1)^4 l - 24 \cos(t1) l \sin(t1)^2 - 4 \sin(t1)^2 l^2$$

$$- 24 \sin(t1)^3 l + 16 \sin(t1)^4 + 48 \cos(t1) \sin(t1)^2 + 16 \sin(t1)^2 l + 48 \sin(t1)^3$$

$$+ 72 \cos(t1) \sin(t1) + 24 \sin(t1) l - 16 \sin(t1)^2 - 48 \sin(t1) - 35)) /$$

$$(2 \sin(t1) \cos(t1) l - 4 \cos(t1) \sin(t1) + 6 \sin(t1) - 6 \cos(t1))$$

$$- \frac{1}{2 \sin(t1) \cos(t1) l - 4 \cos(t1) \sin(t1) + 6 \sin(t1) - 6 \cos(t1)} \Bigg), s =$$

$$-(2 \mathit{RootOf}((4 \sin(t1)^2 l^2 - 24 \sin(t1) l - 16 \sin(t1)^2 - 48 \cos(t1) + 88) \_Z^2 + ($$

$$-4 \sin(t1)^2 l + 8 \sin(t1)^2 + 12 \cos(t1) + 12 \sin(t1) - 8) \_Z + 4 \sin(t1)^4 l^2$$

$$- 16 \sin(t1)^4 l - 24 \cos(t1) l \sin(t1)^2 - 4 \sin(t1)^2 l^2 - 24 \sin(t1)^3 l + 16 \sin(t1)^4$$

$$+ 48 \cos(t1) \sin(t1)^2 + 16 \sin(t1)^2 l + 48 \sin(t1)^3 + 72 \cos(t1) \sin(t1) + 24 \sin(t1) l$$

$$- 16 \sin(t1)^2 - 48 \sin(t1) - 35) \sin(t1)^3 l + 2 \mathit{RootOf}((4 \sin(t1)^2 l^2 - 24 \sin(t1) l$$

$$- 16 \sin(t1)^2 - 48 \cos(t1) + 88) \_Z^2 + (-4 \sin(t1)^2 l + 8 \sin(t1)^2 + 12 \cos(t1)$$

$$+ 12 \sin(t1) - 8) \_Z + 4 \sin(t1)^4 l^2 - 16 \sin(t1)^4 l - 24 \cos(t1) l \sin(t1)^2$$

$$- 4 \sin(t1)^2 l^2 - 24 \sin(t1)^3 l + 16 \sin(t1)^4 + 48 \cos(t1) \sin(t1)^2 + 16 \sin(t1)^2 l$$

$$+ 48 \sin(t1)^3 + 72 \cos(t1) \sin(t1) + 24 \sin(t1) l - 16 \sin(t1)^2 - 48 \sin(t1) - 35)$$

$$\sin(t1) \cos(t1)^2 l + 4 \sin(t1)^2 \cos(t1) l^2 - 4 \mathit{RootOf}((4 \sin(t1)^2 l^2 - 24 \sin(t1) l$$

$$- 16 \sin(t1)^2 - 48 \cos(t1) + 88) \_Z^2 + (-4 \sin(t1)^2 l + 8 \sin(t1)^2 + 12 \cos(t1)$$

$$+ 12 \sin(t1) - 8) \_Z + 4 \sin(t1)^4 l^2 - 16 \sin(t1)^4 l - 24 \cos(t1) l \sin(t1)^2$$

$$- 4 \sin(t1)^2 l^2 - 24 \sin(t1)^3 l + 16 \sin(t1)^4 + 48 \cos(t1) \sin(t1)^2 + 16 \sin(t1)^2 l$$

$$+ 48 \sin(t1)^3 + 72 \cos(t1) \sin(t1) + 24 \sin(t1) l - 16 \sin(t1)^2 - 48 \sin(t1) - 35)$$

$$\sin(t1)^3 - 4 \mathit{RootOf}((4 \sin(t1)^2 l^2 - 24 \sin(t1) l - 16 \sin(t1)^2 - 48 \cos(t1) + 88) \_Z^2$$

$$+ \left(-4 \sin(t1)^2 l + 8 \sin(t1)^2 + 12 \cos(t1) + 12 \sin(t1) - 8\right) \_Z + 4 \sin(t1)^4 l^2$$
$$- 16 \sin(t1)^4 l - 24 \cos(t1) l \sin(t1)^2 - 4 \sin(t1)^2 l^2 - 24 \sin(t1)^3 l + 16 \sin(t1)^4$$
$$+ 48 \cos(t1) \sin(t1)^2 + 16 \sin(t1)^2 l + 48 \sin(t1)^3 + 72 \cos(t1) \sin(t1) + 24 \sin(t1) l$$
$$- 16 \sin(t1)^2 - 48 \sin(t1) - 35\Big) \sin(t1) \cos(t1)^2 - 8 \cos(t1) l \sin(t1)^2$$
$$- 12\, RootOf\Big(\left(4 \sin(t1)^2 l^2 - 24 \sin(t1) l - 16 \sin(t1)^2 - 48 \cos(t1) + 88\right) \_Z^2 + \Big($$
$$-4 \sin(t1)^2 l + 8 \sin(t1)^2 + 12 \cos(t1) + 12 \sin(t1) - 8\Big) \_Z + 4 \sin(t1)^4 l^2$$
$$- 16 \sin(t1)^4 l - 24 \cos(t1) l \sin(t1)^2 - 4 \sin(t1)^2 l^2 - 24 \sin(t1)^3 l + 16 \sin(t1)^4$$
$$+ 48 \cos(t1) \sin(t1)^2 + 16 \sin(t1)^2 l + 48 \sin(t1)^3 + 72 \cos(t1) \sin(t1) + 24 \sin(t1) l$$
$$- 16 \sin(t1)^2 - 48 \sin(t1) - 35\Big) \sin(t1)^2 - 12\, RootOf\Big(\Big(4 \sin(t1)^2 l^2 - 24 \sin(t1) l$$
$$- 16 \sin(t1)^2 - 48 \cos(t1) + 88\Big) \_Z^2 + \Big(-4 \sin(t1)^2 l + 8 \sin(t1)^2 + 12 \cos(t1)$$
$$+ 12 \sin(t1) - 8\Big) \_Z + 4 \sin(t1)^4 l^2 - 16 \sin(t1)^4 l - 24 \cos(t1) l \sin(t1)^2$$
$$- 4 \sin(t1)^2 l^2 - 24 \sin(t1)^3 l + 16 \sin(t1)^4 + 48 \cos(t1) \sin(t1)^2 + 16 \sin(t1)^2 l$$
$$+ 48 \sin(t1)^3 + 72 \cos(t1) \sin(t1) + 24 \sin(t1) l - 16 \sin(t1)^2 - 48 \sin(t1) - 35\Big)$$
$$\cos(t1)^2 + 12 \sin(t1)^2 l - 24 \sin(t1) \cos(t1) l + 4\, RootOf\Big(\Big(4 \sin(t1)^2 l^2 - 24 \sin(t1) l$$
$$- 16 \sin(t1)^2 - 48 \cos(t1) + 88\Big) \_Z^2 + \Big(-4 \sin(t1)^2 l + 8 \sin(t1)^2 + 12 \cos(t1)$$
$$+ 12 \sin(t1) - 8\Big) \_Z + 4 \sin(t1)^4 l^2 - 16 \sin(t1)^4 l - 24 \cos(t1) l \sin(t1)^2$$
$$- 4 \sin(t1)^2 l^2 - 24 \sin(t1)^3 l + 16 \sin(t1)^4 + 48 \cos(t1) \sin(t1)^2 + 16 \sin(t1)^2 l$$
$$+ 48 \sin(t1)^3 + 72 \cos(t1) \sin(t1) + 24 \sin(t1) l - 16 \sin(t1)^2 - 48 \sin(t1) - 35\Big)$$
$$\sin(t1) + 4\, RootOf\Big(\left(4 \sin(t1)^2 l^2 - 24 \sin(t1) l - 16 \sin(t1)^2 - 48 \cos(t1) + 88\right) \_Z^2$$
$$+ \left(-4 \sin(t1)^2 l + 8 \sin(t1)^2 + 12 \cos(t1) + 12 \sin(t1) - 8\right) \_Z + 4 \sin(t1)^4 l^2$$
$$- 16 \sin(t1)^4 l - 24 \cos(t1) l \sin(t1)^2 - 4 \sin(t1)^2 l^2 - 24 \sin(t1)^3 l + 16 \sin(t1)^4$$
$$+ 48 \cos(t1) \sin(t1)^2 + 16 \sin(t1)^2 l + 48 \sin(t1)^3 + 72 \cos(t1) \sin(t1) + 24 \sin(t1) l$$
$$- 16 \sin(t1)^2 - 48 \sin(t1) - 35\Big) \cos(t1) + 24 \cos(t1) \sin(t1) - 37 \sin(t1)$$
$$+ 35 \cos(t1)\Big) \Big/ \Big(2 \Big(2\, RootOf\Big(\Big(4 \sin(t1)^2 l^2 - 24 \sin(t1) l - 16 \sin(t1)^2 - 48 \cos(t1)$$
$$+ 88\Big) \_Z^2 + \left(-4 \sin(t1)^2 l + 8 \sin(t1)^2 + 12 \cos(t1) + 12 \sin(t1) - 8\right) \_Z$$
$$+ 4 \sin(t1)^4 l^2 - 16 \sin(t1)^4 l - 24 \cos(t1) l \sin(t1)^2 - 4 \sin(t1)^2 l^2 - 24 \sin(t1)^3 l$$
$$+ 16 \sin(t1)^4 + 48 \cos(t1) \sin(t1)^2 + 16 \sin(t1)^2 l + 48 \sin(t1)^3 + 72 \cos(t1) \sin(t1)$$
$$+ 24 \sin(t1) l - 16 \sin(t1)^2 - 48 \sin(t1) - 35\Big) \sin(t1)^3 l + 2\, RootOf\Big(\Big(4 \sin(t1)^2 l^2$$
$$- 24 \sin(t1) l - 16 \sin(t1)^2 - 48 \cos(t1) + 88\Big) \_Z^2 + \Big(-4 \sin(t1)^2 l + 8 \sin(t1)^2$$
$$+ 12 \cos(t1) + 12 \sin(t1) - 8\Big) \_Z + 4 \sin(t1)^4 l^2 - 16 \sin(t1)^4 l - 24 \cos(t1) l \sin(t1)^2$$
$$- 4 \sin(t1)^2 l^2 - 24 \sin(t1)^3 l + 16 \sin(t1)^4 + 48 \cos(t1) \sin(t1)^2 + 16 \sin(t1)^2 l$$
$$+ 48 \sin(t1)^3 + 72 \cos(t1) \sin(t1) + 24 \sin(t1) l - 16 \sin(t1)^2 - 48 \sin(t1) - 35\Big)$$

$$\sin(t1)\cos(t1)^2\, l - 4\, RootOf\big((4\sin(t1)^2 l^2 - 24\sin(t1)\, l - 16\sin(t1)^2 - 48\cos(t1)$$

$$+ 88)\,\_Z^2 + (-4\sin(t1)^2\, l + 8\sin(t1)^2 + 12\cos(t1) + 12\sin(t1) - 8)\,\_Z$$

$$+ 4\sin(t1)^4 l^2 - 16\sin(t1)^4\, l - 24\cos(t1)\, l\sin(t1)^2 - 4\sin(t1)^2 l^2 - 24\sin(t1)^3\, l$$

$$+ 16\sin(t1)^4 + 48\cos(t1)\sin(t1)^2 + 16\sin(t1)^2\, l + 48\sin(t1)^3 + 72\cos(t1)\sin(t1)$$

$$+ 24\sin(t1)\, l - 16\sin(t1)^2 - 48\sin(t1) - 35\big)\sin(t1)^3 - 4\, RootOf\big((4\sin(t1)^2 l^2$$

$$- 24\sin(t1)\, l - 16\sin(t1)^2 - 48\cos(t1) + 88)\,\_Z^2 + (-4\sin(t1)^2\, l + 8\sin(t1)^2$$

$$+ 12\cos(t1) + 12\sin(t1) - 8)\,\_Z + 4\sin(t1)^4 l^2 - 16\sin(t1)^4\, l - 24\cos(t1)\, l\sin(t1)^2$$

$$- 4\sin(t1)^2 l^2 - 24\sin(t1)^3\, l + 16\sin(t1)^4 + 48\cos(t1)\sin(t1)^2 + 16\sin(t1)^2\, l$$

$$+ 48\sin(t1)^3 + 72\cos(t1)\sin(t1) + 24\sin(t1)\, l - 16\sin(t1)^2 - 48\sin(t1) - 35\big)$$

$$\sin(t1)\cos(t1)^2 - 6\, RootOf\big((4\sin(t1)^2 l^2 - 24\sin(t1)\, l - 16\sin(t1)^2 - 48\cos(t1)$$

$$+ 88)\,\_Z^2 + (-4\sin(t1)^2\, l + 8\sin(t1)^2 + 12\cos(t1) + 12\sin(t1) - 8)\,\_Z$$

$$+ 4\sin(t1)^4 l^2 - 16\sin(t1)^4\, l - 24\cos(t1)\, l\sin(t1)^2 - 4\sin(t1)^2 l^2 - 24\sin(t1)^3\, l$$

$$+ 16\sin(t1)^4 + 48\cos(t1)\sin(t1)^2 + 16\sin(t1)^2\, l + 48\sin(t1)^3 + 72\cos(t1)\sin(t1)$$

$$+ 24\sin(t1)\, l - 16\sin(t1)^2 - 48\sin(t1) - 35\big)\sin(t1)^2 - 6\, RootOf\big((4\sin(t1)^2 l^2$$

$$- 24\sin(t1)\, l - 16\sin(t1)^2 - 48\cos(t1) + 88)\,\_Z^2 + (-4\sin(t1)^2\, l + 8\sin(t1)^2$$

$$+ 12\cos(t1) + 12\sin(t1) - 8)\,\_Z + 4\sin(t1)^4 l^2 - 16\sin(t1)^4\, l - 24\cos(t1)\, l\sin(t1)^2$$

$$- 4\sin(t1)^2 l^2 - 24\sin(t1)^3\, l + 16\sin(t1)^4 + 48\cos(t1)\sin(t1)^2 + 16\sin(t1)^2\, l$$

$$+ 48\sin(t1)^3 + 72\cos(t1)\sin(t1) + 24\sin(t1)\, l - 16\sin(t1)^2 - 48\sin(t1) - 35\big)$$

$$\cos(t1)^2 + 4\, RootOf\big((4\sin(t1)^2 l^2 - 24\sin(t1)\, l - 16\sin(t1)^2 - 48\cos(t1) + 88)\,\_Z^2$$

$$+ (-4\sin(t1)^2\, l + 8\sin(t1)^2 + 12\cos(t1) + 12\sin(t1) - 8)\,\_Z + 4\sin(t1)^4 l^2$$

$$- 16\sin(t1)^4\, l - 24\cos(t1)\, l\sin(t1)^2 - 4\sin(t1)^2 l^2 - 24\sin(t1)^3\, l + 16\sin(t1)^4$$

$$+ 48\cos(t1)\sin(t1)^2 + 16\sin(t1)^2\, l + 48\sin(t1)^3 + 72\cos(t1)\sin(t1) + 24\sin(t1)\, l$$

$$- 16\sin(t1)^2 - 48\sin(t1) - 35\big)\sin(t1) - \sin(t1)\big)\big)\big]\big]$$

In this example we have four revolute junctions:

```
> Ah3 := Matrix([[cos(t4),-sin(t4),l3], [sin(t4), cos(t4), 0], [0, 0, 1]]);
  A32 := Matrix([[cos(t3),-sin(t3),l2], [sin(t3), cos(t3), 0], [0, 0, 1]]);
  A21 := Matrix([[cos(t2),-sin(t2),l1], [sin(t2), cos(t2), 0], [0, 0, 1]]);
  A10 := Matrix([[cos(t1),-sin(t1), 0], [sin(t1), cos(t1), 0], [0, 0, 1]]);
  Conf := MatrixMatrixMultiply(A10, MatrixMatrixMultiply(A21, MatrixMatrixMultiply(A32,
     Ah3)));
```

$$Ah3 := \begin{bmatrix} \cos(t4) & -\sin(t4) & l3 \\ \sin(t4) & \cos(t4) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$A32 := \begin{bmatrix} \cos(t3) & -\sin(t3) & l2 \\ \sin(t3) & \cos(t3) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$A21 := \begin{bmatrix} \cos(t2) & -\sin(t2) & l1 \\ \sin(t2) & \cos(t2) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$A10 := \begin{bmatrix} \cos(t1) & -\sin(t1) & 0 \\ \sin(t1) & \cos(t1) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{aligned} Conf := [&[\cos(t1)\,(\cos(t2)\,(\cos(t3)\cos(t4) - \sin(t3)\sin(t4)) - \sin(t2)\,(\sin(t3)\cos(t4) \\ &+ \cos(t3)\sin(t4))) - \sin(t1)\,(\sin(t2)\,(\cos(t3)\cos(t4) - \sin(t3)\sin(t4)) \\ &+ \cos(t2)\,(\sin(t3)\cos(t4) + \cos(t3)\sin(t4))), \cos(t1)\,(\cos(t2)\,(-\cos(t3)\sin(t4) \\ &- \sin(t3)\cos(t4)) - \sin(t2)\,(\cos(t3)\cos(t4) - \sin(t3)\sin(t4))) - \sin(t1)\,(\sin(t2)\,(\\ &-\cos(t3)\sin(t4) - \sin(t3)\cos(t4)) + \cos(t2)\,(\cos(t3)\cos(t4) - \sin(t3)\sin(t4))), \\ &\cos(t1)\,(\cos(t2)\,(\cos(t3)\,l3 + l2) - \sin(t2)\sin(t3)\,l3 + l1) \\ &- \sin(t1)\,(\sin(t2)\,(\cos(t3)\,l3 + l2) + \cos(t2)\sin(t3)\,l3)], \\ &[\sin(t1)\,(\cos(t2)\,(\cos(t3)\cos(t4) - \sin(t3)\sin(t4)) - \sin(t2)\,(\sin(t3)\cos(t4) \\ &+ \cos(t3)\sin(t4))) + \cos(t1)\,(\sin(t2)\,(\cos(t3)\cos(t4) - \sin(t3)\sin(t4)) \\ &+ \cos(t2)\,(\sin(t3)\cos(t4) + \cos(t3)\sin(t4))), \sin(t1)\,(\cos(t2)\,(-\cos(t3)\sin(t4) \\ &- \sin(t3)\cos(t4)) - \sin(t2)\,(\cos(t3)\cos(t4) - \sin(t3)\sin(t4))) + \cos(t1)\,(\sin(t2)\,(\\ &-\cos(t3)\sin(t4) - \sin(t3)\cos(t4)) + \cos(t2)\,(\cos(t3)\cos(t4) - \sin(t3)\sin(t4))), \\ &\sin(t1)\,(\cos(t2)\,(\cos(t3)\,l3 + l2) - \sin(t2)\sin(t3)\,l3 + l1) \\ &+ \cos(t1)\,(\sin(t2)\,(\cos(t3)\,l3 + l2) + \cos(t2)\sin(t3)\,l3)], \\ &[0, 0, 1]] \end{aligned} \tag{4}$$