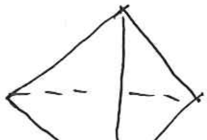
Polyhedra

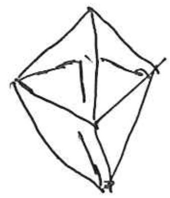
A regular polyhedron is a convex body whose faces ar regular polygon of the same type and whose vertex configurations are all the same.

The angle around a vertex is smaller than 2π.

$$k \frac{(n-2)\pi}{n} \underset{\neq}{\leq} 2\pi \quad ; \quad (k-2)(n-2) < 4 \quad k, n \geq 3$$

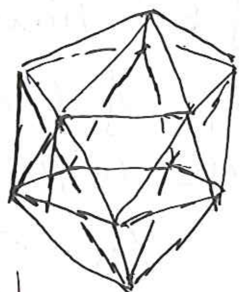
$n = 3$; $k = 3$ tetrahedron

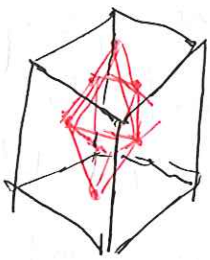




$n = 3$ $k = 4$ octahedron

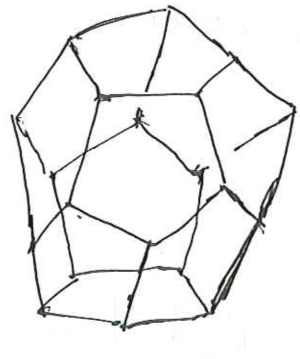
$n = 3$ $k = 5$ icosahedron



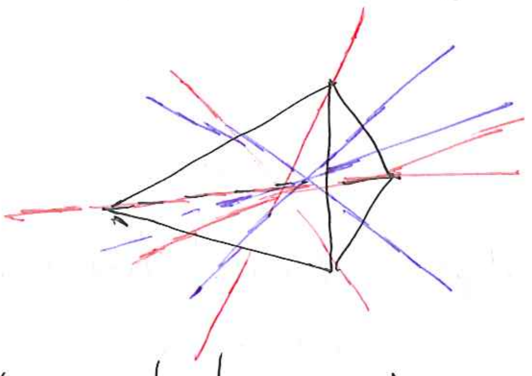


$n = 4$ $k = 3$ cube (hexahedron)

$n = 5$ $k = 3$ dodecahedron



Rotations of a tetrahedron



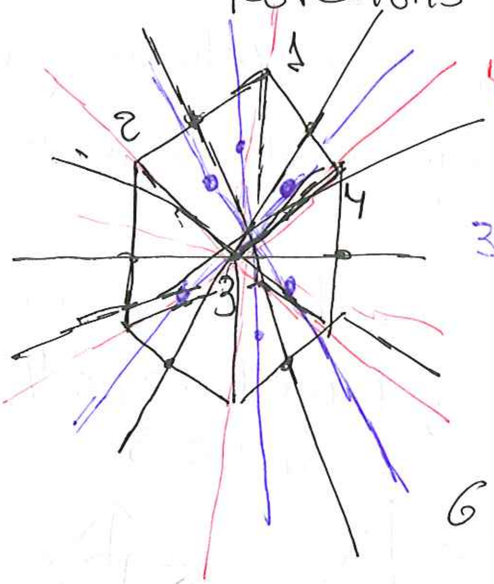
4 vertices (midpoints) : 4×2 rotations of $2\pi/3$-angle.

3 pairs of faced edges 3 rotations of π-angle

The identity.

The rotations of a regular tetrahedron form the group $A_4 = \langle \rho_1, \rho_2 \mid \rho_1^3 = \rho_2^2 = (\rho_1 \rho_2)^3 = Id \rangle$

Rotations of the cube (octahedron)



4 pairs of opposite vertices
4×2 rotations of angle $\frac{2\pi}{3}$

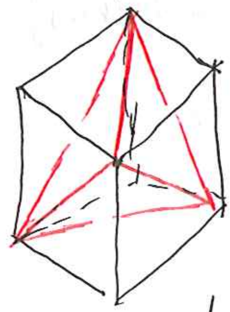
3 pairs of faced faces
6 rotations of angle $\pi/2$
3 rotations of angle π

6 pairs of faced (opposite) edges
6 rotations of angle π

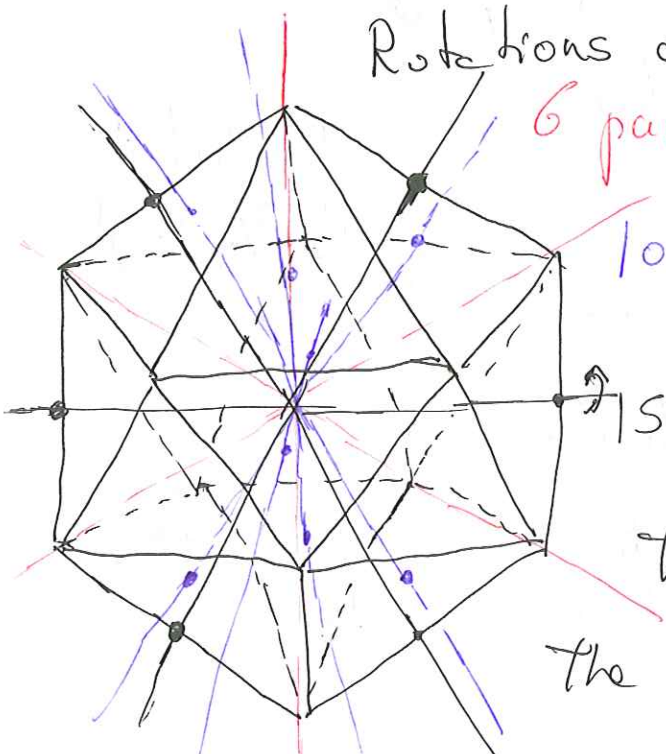
The identity

The group of rotations is Σ_4 (to see this we must thing only of the vertices marked 1, 2, 3, 4 and see what happens with them)

From Algebra $A_4 \leq \Sigma_4$; we see the tetrahedron inside the cube.



Rotations of the dodecahedron (icosahedron)



6 pairs of vertices $= 6 \times 4$ rotations of angles $\frac{2\pi}{5}$ (and $\frac{4\pi}{5}$)

10 pair of faces $= 10 \times 2$ rotations of angle $2\pi/3$

15 pairs of edges $=$ 15 rotations of angle π

The identity
$\overline{\text{60 rotations}}$

The group of rotations is A_5

Frieze patterns

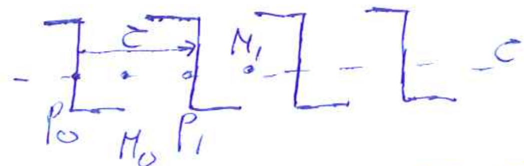
- Frieze are equivalent if they have the same developing-group = frieze group
- shortest translation τ: $F = \langle \tau \rangle$, axis c.
- Symmetries of the frieze-group are isometries

p { centre of a half-turn if G contains half-tr
$p \perp c$ — G contains reflection in p, $p \perp c$
point in c

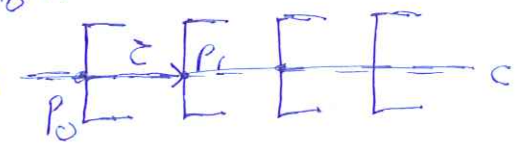
$P_0 = P$ s $\quad P_n = \tau^n(P) \quad M_0 = \text{midpoint } P_0P_1 \quad M_n = \tau^n(M_0) = \text{mid } P_nP_{n+1}$

$g_1 = \langle \tau \rangle$

$g_2 = \langle H_p, \tau \rangle \qquad \tau H_p = H_M.$



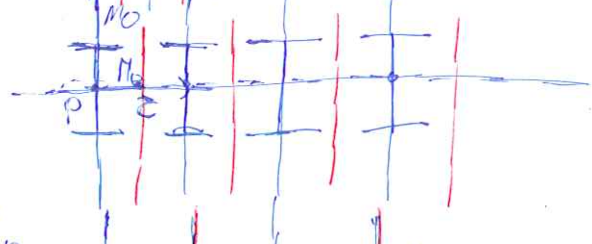
$g_3 = \langle \tau, R_c \rangle \qquad \tau^n R_c = \text{glide reflection}$



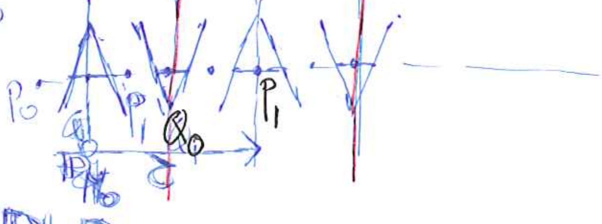
Horizontal reflection

$g_4 = \langle \tau, R_p \rangle \qquad \tau R_p = R_{M_0}$

$g_5 = \langle \tau, R_c, R_p \rangle \qquad H_p = R_p R_c$
$\tau R_p = R_{M_0}$



$g_6 = \langle \tau, H_p, R_q \rangle \quad q \perp c, \; q \neq p$
$Q = q \cdot c$ is a midpoint, for instance M_0



$g_7 = \langle \sigma \rangle \qquad \sigma^2 = \tau$

Symmetries of wallpaper.

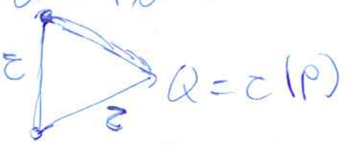
G = translations in two independent directions and symmetry leaving fixed points (point group)

- The rotation order must be 2, 3, 4 or 6 (keeping distances)

1) $n \leq 6$ Assume a rotation $R_{P,\theta}$ of order $n \geq 7$ and centre P

τ = translations of shortest length

$Q' = R_{P,\theta}(Q)$ $\qquad R_P R_Q R_P^{-1} = R_{Q'}$ rotation with centre Q'



$\tau R_P \tau^{-1} = R_{Q\,\theta}$ rotation with centre Q.

P In the circle with centre P and radius $|\tau|$

$|\overline{QQ'}| < |\tau|$ Contradiction since $T_{\overrightarrow{QQ'}}$ is a symmetry of the wallpaper

2) $n \neq 5$ Assume a rotation $R_{P,\frac{2\pi}{5}}$ with centre P

τ = shortest translation

$Q' = R_P(Q)$

$R_Q = \tau R_P \tau^{-1}$ with centre Q

$R_{Q'} = R_P R_Q R_P^{-1}$ with centre Q'.

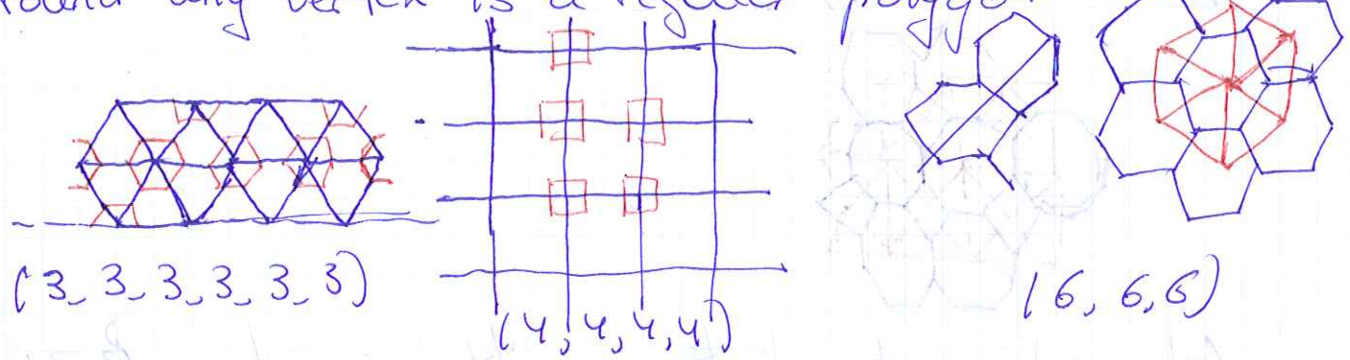
$R_{P'} = R_Q^{-1} R_P R_Q$ with centre P'

$T_{\overrightarrow{PQ'}}$ is another symmetry of the wallpaper with length of translation smaller than $|\tau|$.

Contradiction

Tessellations of the plane

Regular tessellation consists of regular polygon of the same kind in such a way that the "star" around any vertex is a regular polygon



$(3,3,3,3,3,3)$ $(4,4,4,4)$ $(6,6,6)$

Around each vertex 2π rad : $k\,\pi\,\frac{(n-2)}{n} = 2\pi$

$$k(n-2) = 2n \quad k \geq 3 \Rightarrow n \leq 6$$

$n=6,\ k=3$; $n=4,\ k=4$; $n=3,\ k=6$.

Signature $(\overbrace{n, --, n}^{k})$

Semiregular tessellation : consists of different kind of regular polygon, but all the stars around vertices are congruent : $\sum_1^k \left(\frac{n_i-2}{n_i}\right)\pi = 2\pi$; $\sum_1^k \left(1-\frac{2}{n_i}\right) = 2$

$$\frac{k-2}{2} = \sum_1^k \frac{1}{n_i} \qquad n_i \geq 3$$

$k=3$ $\frac{1}{n_1}+\frac{1}{n_2}+\frac{1}{n_3}=\frac{1}{2}$ signatures $(6,6,6)$ $(4,6,12)$ $(4,8,8)$ $(3,12,12)$

Why not $(3,8,24)$

$k=4$; $\sum_1^4 \frac{1}{n_i} = 1$ signatures $(4,4,4,4)$ $(3,6,3,6)$ $(3,4,6,4)$ } geometry, orientation

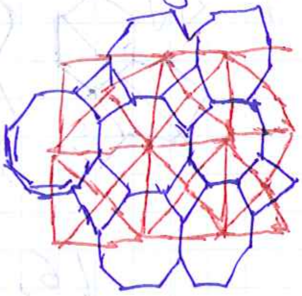
$k=5$ $\sum_1^5 \frac{1}{n_i} = 3/2$ signatures $(3,3,3,3,6)$ $(3,3,3,4,4)$ $(3,3,4,3,4)$

$k=6$ $\sum_1^6 \frac{1}{n_i} = 2$ signature $(6,6,6,6,6,6)$

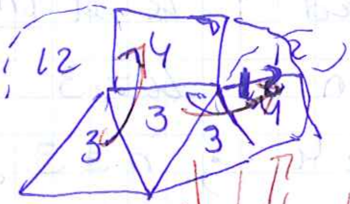
Dual tessellation $\tilde{T}$ of a tessellation T has as vertices the midpoints of the polygons in T and the edges are the edges joining the midpoints of two polygons of T with a common edge in T

(4, 4, 4, 4) autodual, the dual of (6,6,6) is (3,3,3,3,3,3)

The dual of (4, 8, 8)



Why not a tessellation with signature (3,3,4,12)??



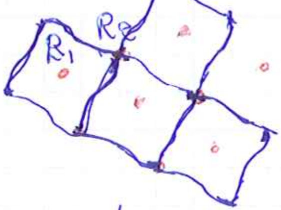
impossible

it should be a 12-gon

it should be a square following the orientation.

Periodic

- Take a motiv and develope it by using a group (instructions to develope the motiv)



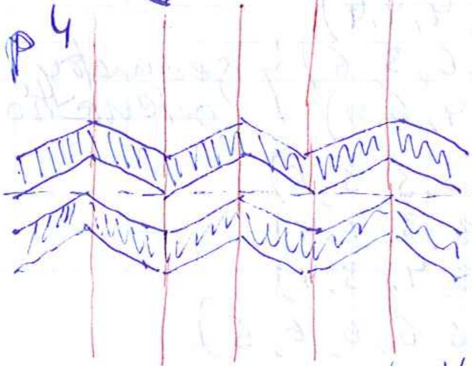
∘ = rotation centres

R_1 and R_2 generate the group.

| reflection axes

p4

pgm



--- glide-reflection axis

They are 17 essentially different ways of ~~...~~ developping (Pólya, Niggli 1924) Nowacki (1954) showed that it was ~~essentia~~ exactly distinct.

ekvationer

$$\frac{1}{n_1}+\frac{1}{n_2}+\frac{1}{n_3}=\frac{1}{2}$$

n_1	n_2	n_3	n_4	n_5	n_6	radnr
6	6	6				1
5	5	10				2
4	5	20				3
4	6	12				4
4	8	8				5
3	7	42				6
3	8	24				7
3	9	18				8
3	10	15				9
3	12	12				10

$$\frac{1}{n_1}+\frac{1}{n_2}+\frac{1}{n_3}+\frac{1}{n_4}=1$$

n_1	n_2	n_3	n_4	n_5	n_6	radnr
4	4	4	4			11
3	3	4	12			12
3	~~3~~ 6	~~6~~ 3	6			13
3	4	~~4~~ 6	~~6~~ 4			14

$$\frac{1}{n_1}+\frac{1}{n_2}+\frac{1}{n_3}+\frac{1}{n_4}+\frac{1}{n_5}=\frac{3}{2}$$

n_1	n_2	n_3	n_4	n_5	n_6	radnr
3	3	3	3	6		15
3	3	3	4	4		16
3	3	4	3	4		

$$\frac{1}{n_1}+\frac{1}{n_2}+\frac{1}{n_3}+\frac{1}{n_4}+\frac{1}{n_5}+\frac{1}{n_6}=2$$

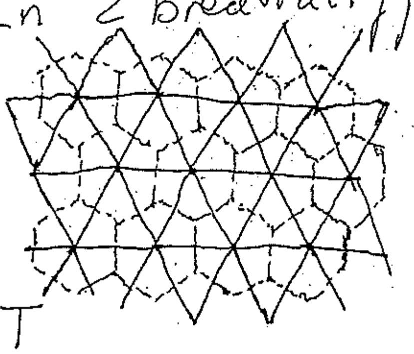
n_1	n_2	n_3	n_4	n_5	n_6	radnr
3	3	3	3	3	3	17

Med **hörnkonfiguration** menas den ordningsföljd som polygonerna har runt ett hörn. Om hörnkonfigurationen är olika vid olika hörn är inte tesseleringen semireguljär.

Radnr 1, 11 och 17 ger de reguljära lösningarna som vi redan visat.

Radnr 4, 5, 10, 13, 14, 15 och 16 ger de semireguljära lösningarna. Observera att lösningen i radnr 16 ger upphov till **två** möjliga hörnkonfigurationer. Vi har således totalt 8 möjliga semireguljära tesselleringar. Samtliga visas i figur6 på nästa sida.

Dual tesselleringen $\bar{T}$ till en given tesselleringen T är den som har som hörn tyngdpunkterna till månghörningar i T och kantér är alla förbindelser mellan 2 bredvidliggande tyngdpunkterna.



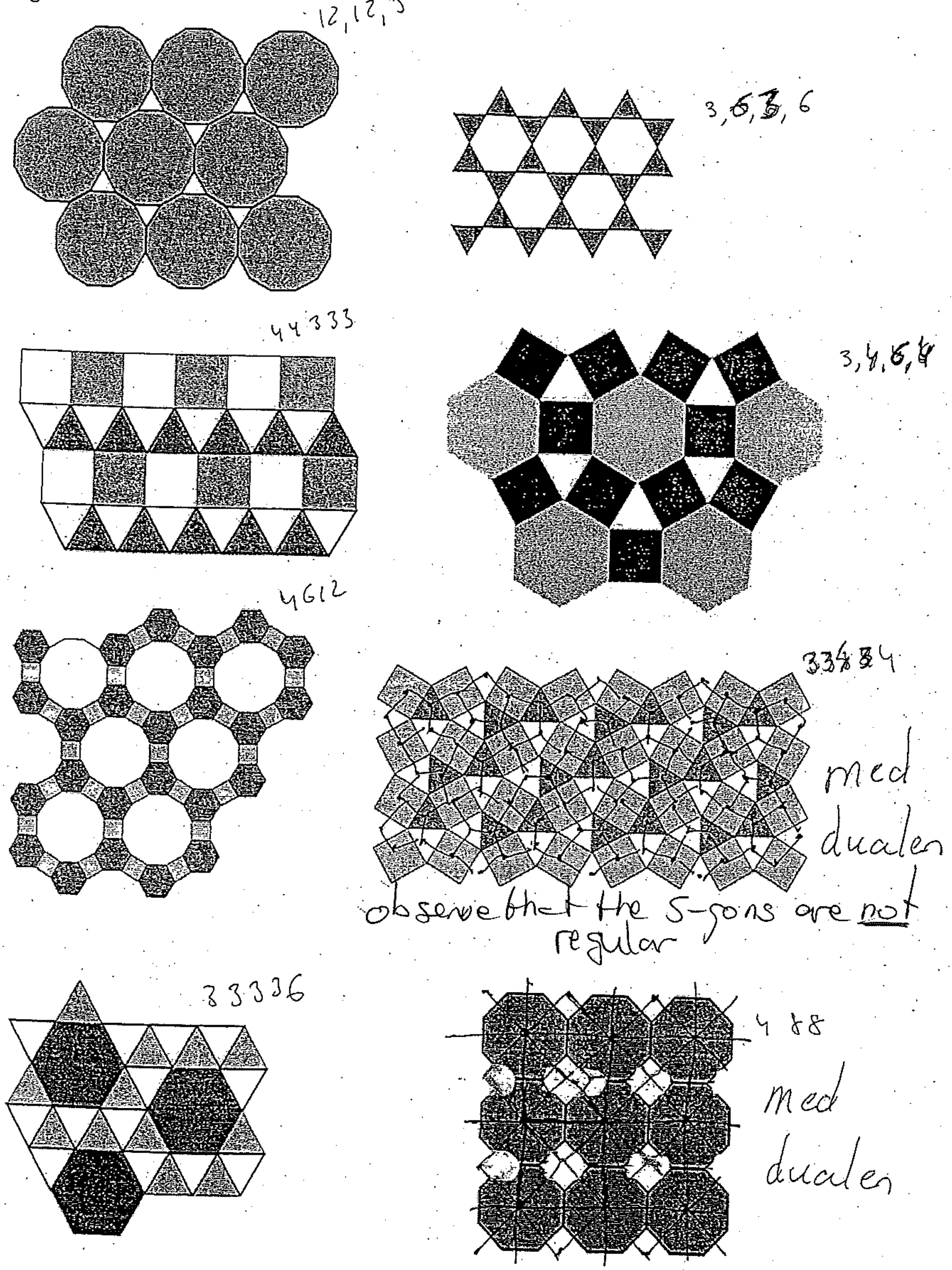
Dual $\bar{T}=(3,3,3,3,3,3)$ till $T=(6,6,6)$

- - - T

— $\bar{T}$

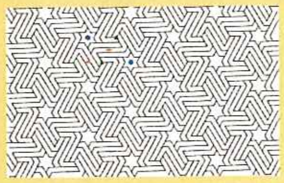
De åtta semireguljära tesselleringarna av planet.

figur 6

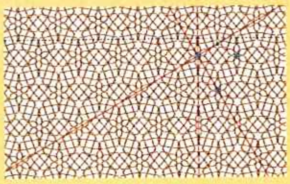


De 17 planära kristallografiska grupper i Alhambra

Grupper med rotationer av vinkel 60°

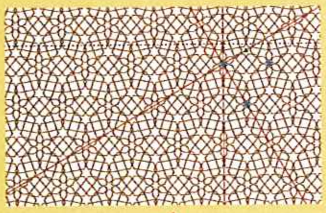


p6

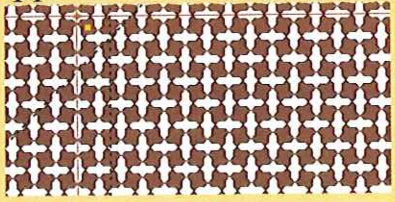


p6m

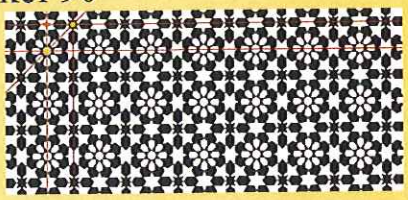
Grupper med rotationer av vinkel 90°



p4

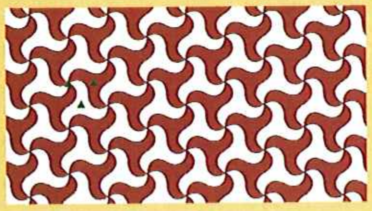


p4g

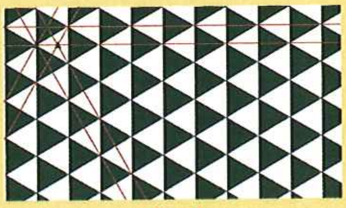


p4m

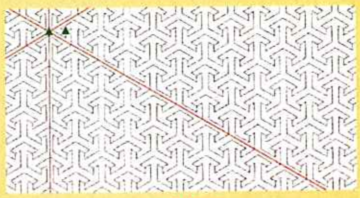
Grupper med rotationer av vinkel 120°



p3

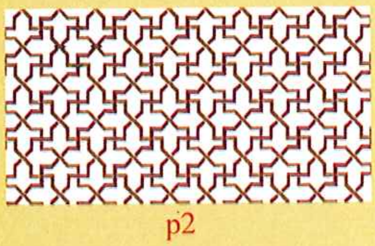


p3m1

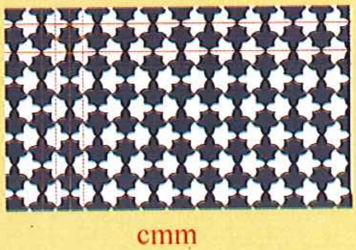


p31m

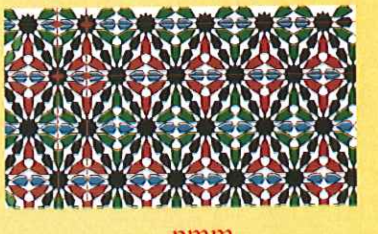
Grupper med rotationer av vinkel 180°



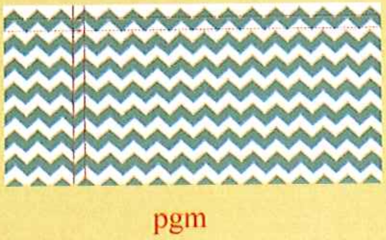
p2



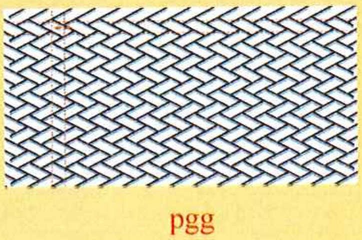
cmm



pmm

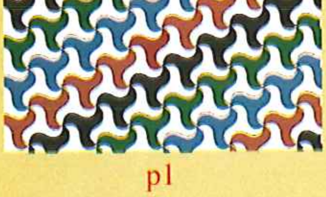


pgm

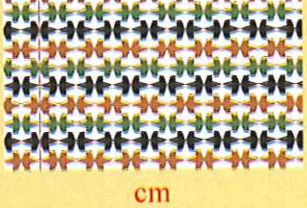


pgg

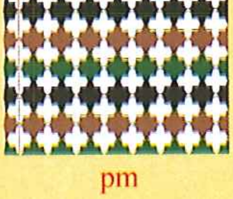
Grupper utan rotationer



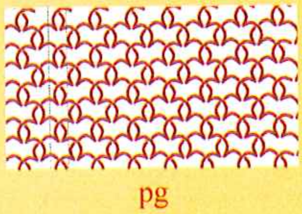
p1



cm



pm



pg

Penrose's Tessellations.

It has local 5-symmetry and it is NOT periodic