Isometries of $\mathbb{R}^3$. Animation and Quaternions

Remember $\mathbb{H} = \{\text{quaternions } q = \{(a, \underset{q}{x}) / a \in \mathbb{R}, x \in \mathbb{R}^3\}$

Pure quaternions $\mathbb{H}_0 = \{(0, x) = x / x \in \mathbb{R}^3\}$

We can describe ~~the~~ reflection in the plane with equation $n x = 0$ $\quad n[n_1, n_2, n_3, 0]$ as follows: consider the quaternion

$n = (0, \mathbb{n})$
$p = (0, \mathbb{p})$ $\Big\}$ $S_n(\mathbb{p}) = -n p n^{-1}$

- Control: invariant points $S_n(x) = -n x n^{-1} = x$ or $-nx = xn$. This is true iff $x \perp \mathbb{n}$

Now, write $x = \mathbb{p} + \lambda \mathbb{n}$ with $\mathbb{p} \perp \mathbb{n}$

$S_n(x) = -n(p + \lambda n) n^{-1} = -n p n^{-1} - \lambda n = \mathbb{p} - \lambda \mathbb{n}$

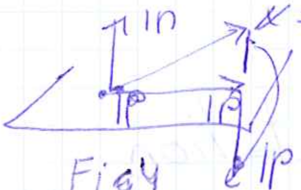


Fig 4

Example. Reflection in the $x_1 x_3$-plane $\mathbb{n} = j$

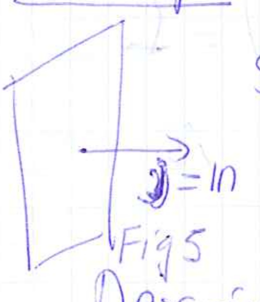


Fig 5

$S(x) = -j x j^{-1} = -j x (-j) = j x j$

$x = (0, x)$

Description of rotations with axes through O

Every rotation is the product of two reflections (in this case in planes through O)

Consider $R = S_{q_2} S_{q_1}$ with $|q_1| = |q_2| = 1$

$q_i = (0, q_i)$

$x = (0, x)$

$R(x) = -q_2 ((-1) q_1 x q_1^{-1}) q_2^{-1}$

$= (q_2 q_1) x (q_2 q_1)^{-1}$ with $|q_2 q_1| = 1$ so $(q_2 q_1)^{-1} = \overline{q_2 q_1}$

we call $\underline{q = q_2 q_1}$

Rotation of axis $\mathbb{n}$ and angle (counterclockwise) θ is given by $R_{(\mathbb{n},\theta)}$ ~~$(\cos\frac{1}{2}\theta, \sin\frac{1}{2}$~~

$$R_q(\mathbf{x}) = q\, x\, q^{-1} \quad \text{with}$$

$x = (0, \mathbf{x})$, $q = (\cos\frac{1}{2}\theta, \sin\frac{1}{2}\theta\, \mathbb{n})$ with $|\mathbb{n}| = 1$

and in the direction of $\mathbb{n}$.

<u>Example</u> $R_j = -j\, x\, j$

$j = (\cos\pi/2, \sin\pi/2\,(0,1,0))$. R_j is a rotation of $90°$ in the x_2-axis

(A typical singularity when representing rotations with Euler-angles)

- The product of two rotations is a rotation

R_{q_1} and R_{q_2} $\quad q_1 = (\cos\frac{1}{2}\theta, \sin(\frac{1}{2}\theta)\mathbb{n}_1)$, ~~$q_2 = (\cos\frac{1}{2}t$~~,

$q_2 = (\cos\frac{1}{2}t, \sin(\frac{1}{2}t)\mathbb{n}_2)$; $|\mathbb{n}_1| = |\mathbb{n}_2| = 1$

$R_{q_2} R_{q_1}(\mathbf{x}) = q_2 q_1 x\, \bar{q}_1 \bar{q}_2 = q_2 q_1\, x\, \overline{q_2 q_1}$ is

a rotation given by $q = q_2\, q_1 =$

$= (\cos\frac{1}{2}t, \sin\frac{1}{2}t\,\mathbb{n}_2)(\cos\frac{1}{2}\theta, \sin\frac{1}{2}\theta\,\mathbb{n}_1) =$

$$= \Big(\underbrace{\cos\tfrac{1}{2}t\cos\tfrac{1}{2}\theta - \sin\tfrac{1}{2}t\sin\tfrac{1}{2}\theta\,\mathbb{n}_1\cdot\mathbb{n}_2}_{\cos(\frac{1}{2}\alpha)},$$

$$, \underbrace{\cos\tfrac{1}{2}\theta\sin\tfrac{1}{2}t\,\mathbb{n}_2 + \cos\tfrac{1}{2}t\sin\tfrac{1}{2}\theta\,\mathbb{n}_1 + \sin\tfrac{1}{2}t\sin\tfrac{1}{2}\theta\,(\mathbb{n}_2\times\mathbb{n}_1)}_{\sin(\frac{1}{2}\alpha)\,\mathbb{n}}\Big)$$

Example i) Rotation around the direction $(-4, 2, 4)$ and angle $\pi/3$ (positive oriented) is given by R_q with $q = \left(\frac{\sqrt{3}}{2}, \frac{1}{6}(-2,1,2)\right)$

$$R_q(x) = \left(\frac{\sqrt{3}}{2} + -\frac{1}{3}i + \frac{1}{6}j + \frac{1}{3}k\right)(x_1 i + x_2 j + x_3 k)\left(\frac{\sqrt{3}}{2} + \frac{1}{3}i - \frac{1}{6}j - \frac{1}{3}k\right)$$

ii) Product of rotation given by quaternions

$$q_1 = \left(1/5, \frac{2\sqrt{6}}{5}\left(\frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}}, \frac{1}{\sqrt{6}}\right)\right) \text{ and } q_2 = \left(2/3, \frac{\sqrt{5}}{3}\left(0, \frac{1}{\sqrt{5}}, 2/\sqrt{5}\right)\right)$$

$$\cos \theta_1/2 = 1/5, \quad \cos \theta_2/2 = 2/3$$

$$\cos\frac{\alpha}{2} = \frac{1}{5}\cdot\frac{2}{3} - \begin{pmatrix} 2/5 \\ 4/5 \\ 2/5 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 1/3 \\ 2/3 \end{pmatrix} = \frac{2}{15} - \frac{8}{15} = -\frac{6}{15}$$

$$\sin\frac{\alpha}{2} = \frac{\sqrt{189}}{15} = \frac{\sqrt{21}}{5}$$

vectorial part is $\frac{1}{5}\begin{pmatrix} 0 \\ 1/3 \\ 2/3 \end{pmatrix} + \frac{2}{3}\begin{pmatrix} 2/5 \\ 4/5 \\ 2/5 \end{pmatrix} + \begin{pmatrix} 0 \\ 1/3 \\ 2/3 \end{pmatrix} \times \begin{pmatrix} 2/5 \\ 4/5 \\ 2/5 \end{pmatrix} = \begin{pmatrix} -2/15 \\ 13/15 \\ 4/15 \end{pmatrix}$

$$\frac{\sqrt{21}}{5}\,n = \begin{pmatrix} -2/15 \\ 13/15 \\ 4/15 \end{pmatrix}; \quad n = \frac{1}{3\sqrt{21}}(-2, 13, 4)$$

$$q = \left(-6/15, \frac{\sqrt{21}}{5}(-2, 13, 4)\right); \quad q = q_2 q_1$$

Exercise 1) Show that given $p = (0, \mathbb{p})$ and $q = (0, \mathbb{q})$, $pq = qp$ iff $\mathbb{p} \perp \mathbb{q}$

2) As in example ii) above, what is the axis of the rotation $R_{q_1} R_{q_2}$??

Application: Animation, rotation

The "~~easiest~~ fastest" way of seeing the change of orientation of a body is by a rotation in 3D (think of the different orientation as two points on S^3)

The conventional way is by using a set of Euler angles $\{t, \phi, \theta\}$ (first problem, to choose the order)

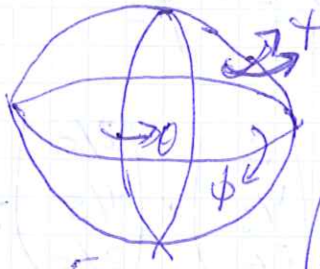


Fig 6

We can express every rotation as

$$\begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} \cos\phi & 0 & \sin\phi \\ 0 & 1 & 0 \\ -\sin\phi & 0 & \cos\phi \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos t & -\sin t \\ 0 & \sin t & \cos t \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix}$$

We have choosen order first around x-axis, then y-axis and finally z-axis.

Observe that, if, for instance ϕ is 90° (or -90°) then we have

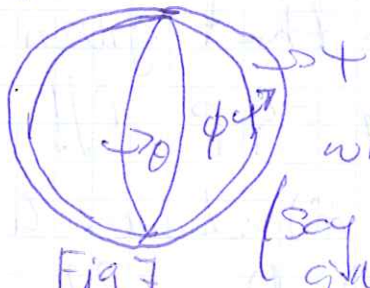


Fig 7

and we do not know what to do (say what instructions to give a camera)

This a singularity is called the <u>gimbal lock</u>

Advantage of using quaternions: No singularities

Also computational advantage:

	Quat	Euler
Representation of rotations:	4 elem.	9 elem

We can do an animation by interpolating the start and final position (orientation) of a body. Remember each position of the body is a rotation in 3D given by a quaternion

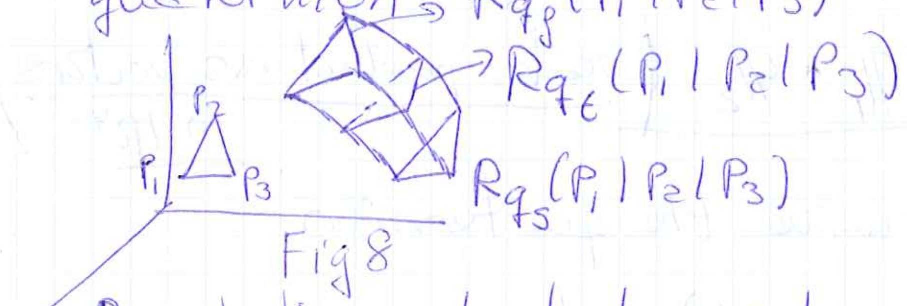


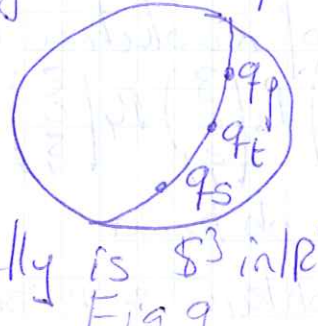
Fig 8

Orientation at start point gives of a rotation with quaternion q_s

Orientation at final point gives of a rotation with quaternion q_f

Orientation at intermediate point $0 \leq t \leq 1$ gives of a rotation with quaternion q_t

We have seen that the set of all unit quaternions form the unit sphere S^3 in $\mathbb{R}^4$. So q_s and q_f are two points in S^3. A path from q_s to q_f is a great circle in S^3



(really is S^3 in $\mathbb{R}^4$)
Fig 9

Any intermediate quaternion q_t will be on that great circle. The method is called: <u>spherical linear interpolation</u> (SLERP)

$$SLERP(q_s, q_f, t) = q_1 (q_1^{-1} q_2)^t \qquad 0 \leq t \leq 1$$

with a lot of calculations, we approximate by

$$q_t = SLERP(q_s, q_f, t) = q_s \frac{\sin(1-t)\theta}{\sin\theta} + q_f \frac{\sin(t\theta)}{\sin\theta}$$

where $q_s = (\cos\theta_s, \sin\theta_s\, \mathbf{q}_s)$ $\quad |\mathbf{q}_s| = |\mathbf{q}_f| = 1$

$q_f = (\cos\theta_f, \sin\theta_f, \mathbf{q}_f)$

and $\underline{\cos\theta = \mathbf{q}_1 \cdot \mathbf{q}_2}$ (scalar product as vectors in $\mathbb{R}^4$)

Exercise. Determine the quaternion

$q_t = SLERP(q_s, q_f, t)$ above.

Exercise Determine the rotation with quaternion $q = \left(1/2, \frac{\sqrt{3}}{6}(2,-1,2)\right)$. (The matrix, thanks)

Exercise Figure 8 is wrong. Why??

Exercise Calculate the angle θ in SLERP above when $q_s = \left(\frac{1}{5}\,(2/5, -4/5, 2/5)\right)$ and $q_f = \left(2/3, (0, 1/3, -2/3)\right)$

Rendering Rotation Surfaces

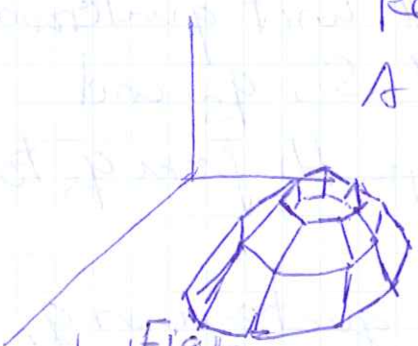


Fig 10

A surface of revolution is obtained by rotating a curve in x_1x_3-plane around the x_3-axis 2π radians. We can have a surface of revolution by rotating the rectangle, say $P_1\begin{pmatrix}1\\0\\2\\1\end{pmatrix}$ $P_2\begin{pmatrix}2\\0\\1\\1\end{pmatrix}$ $P_3\begin{pmatrix}3\\0\\2\\1\end{pmatrix}$ $P_4\begin{pmatrix}2\\0\\3\\1\end{pmatrix}$ through $2\pi j/n$ for $0 \le j \le n$. We obtain $n+1$ instances of the rectangle to obtain a mesh that can be filled to render the surface.

Instances
$$\begin{pmatrix}\cos\frac{2\pi j}{n} & -\sin\frac{2\pi j}{n} & 0 & 0\\ \sin\frac{2\pi j}{n} & \cos\frac{2\pi j}{n} & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{pmatrix}\begin{pmatrix}1 & 2 & 3 & 2\\ 0 & 0 & 0 & 0\\ 2 & 1 & 2 & 3\\ 1 & 1 & 1 & 1\end{pmatrix}$$