Sol to exercises in projective geometry

4.7.4) $l[a,b,c]$: ideal points $l \cdot z$, where $z[0,0,1]$

Equation for ideal(s) point(s) is $\begin{vmatrix} n_1 & n_2 & n_3 \\ a & b & c \\ 0 & 0 & 1 \end{vmatrix} = 0$ $\quad bn_1 - an_2 = 0$

so $I\begin{bmatrix} -b \\ a \\ 0 \end{bmatrix}$. Observe that in fact $v = \begin{pmatrix} -b \\ a \end{pmatrix}$ is the direction vector of the line l, as it should be (the ideal point is the direction of the line)

4.7.5) Equation of the line $P\begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} Q\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ is $\begin{vmatrix} x_1 & 0 & 1 \\ x_2 & 2 & 1 \\ x_3 & 1 & 0 \end{vmatrix} = 0$

$-x_1 + x_2 - 2x_3 = 0$ and coordinates $l[1,-1,2]$

Point of intersection of $2x_2 + x_3 = 0$, $x_1 + x_2 = 0 \Rightarrow P\begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$

$x_1 = -x_2 = \frac{1}{2} x_3$

line joining $2u_2 + u_3 = 0$, $u_1 + u_2 = 0$ i.e. P and Q, so $l[1,-1,2]$

4.7.6) $l[a,b,c]$ and $m[a,b,d]$ $(c \neq d)$ are parallel lines in E^2

$l \cdot m$ = the ideal point of them, i.e. their direction $P\begin{bmatrix} -b \\ a \\ 0 \end{bmatrix}$

(Control if you want with calculations $\begin{vmatrix} n_1 & n_2 & n_3 \\ a & b & c \\ a & b & d \end{vmatrix} = 0 \Leftrightarrow \begin{vmatrix} n_1 & n_2 & n_3 \\ a & b & c \\ 0 & 0 & 1 \end{vmatrix} = 0$

$bn_1 - an_2 = 0 \to P\begin{bmatrix} -b \\ a \\ 0 \end{bmatrix}$)

4.7.8) $P\begin{bmatrix} 2 \\ 3 \\ -2 \end{bmatrix}$, $Q\begin{bmatrix} 1 \\ 2 \\ -4 \end{bmatrix}$ and $R\begin{bmatrix} 0 \\ 1 \\ -6 \end{bmatrix}$ are collinear since

$\left(\begin{array}{cc|c} 1 & 2 & 0 \\ 2 & 3 & 1 \\ -4 & -2 & -6 \end{array}\right) \sim \left(\begin{array}{cc|c} 1 & 2 & 0 \\ 0 & -1 & 1 \\ 0 & 4 & -4 \end{array}\right) \sim \left(\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{array}\right)$ $\quad R = 2Q - P$.

and parameter (coordinate) on the affine line. $\underline{R = -2}$

4.8.1 $s\begin{pmatrix} \lambda_1' \\ \lambda_2' \end{pmatrix} = A\begin{pmatrix} \lambda_1 \\ x_2 \end{pmatrix}$ $\quad P \to P'$ and $Q \to Q'$. Then

$s_1\begin{pmatrix} 1 \\ 0 \end{pmatrix} = A\begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{cases} a_{11} = s_1 \\ a_{12} = 0 \end{cases}$ $\quad s_2\begin{pmatrix} 0 \\ 1 \end{pmatrix} = A\begin{pmatrix} 0 \\ 1 \end{pmatrix} \begin{cases} 0 = a_{21} \\ s_2 = a_{22} \end{cases}$

$A = \begin{pmatrix} s_1 & 0 \\ 0 & s_2 \end{pmatrix}$.

4.8.2 $T\begin{pmatrix}0\\1\end{pmatrix}=\begin{bmatrix}1\\2\end{bmatrix}$, $T\begin{bmatrix}1\\0\end{bmatrix}=\begin{bmatrix}2\\3\end{bmatrix}$, $T\begin{bmatrix}1\\1\end{bmatrix}=\begin{bmatrix}-1\\0\end{bmatrix}$

$s_1\begin{pmatrix}1\\2\end{pmatrix}=A\begin{pmatrix}0\\1\end{pmatrix}$ | $s_2\begin{pmatrix}2\\3\end{pmatrix}=A\begin{pmatrix}1\\0\end{pmatrix}$ | $\begin{pmatrix}-1\\0\end{pmatrix}=\begin{pmatrix}2s_2 & s_1\\3s_2 & 2s_1\end{pmatrix}\begin{pmatrix}1\\1\end{pmatrix}$

$s_1=a_{12}$ | $2s_2=a_{11}$ | $-1=2s_2+s_1$, $0=3s_2+2s_1$ $\Big\{\, 2=-s_2,\ s_1=3$

$2s_1=a_{22}$ | $3s_2=a_{21}$

$s\begin{pmatrix}\lambda_1'\\\lambda_2'\end{pmatrix}=\begin{pmatrix}-4 & 3\\-6 & 6\end{pmatrix}\begin{pmatrix}\lambda_1\\\lambda_2\end{pmatrix}$

4.8.4) Find the invariant elements of the projectivity

$s\begin{pmatrix}\lambda_1\\\lambda_2\end{pmatrix}=\begin{pmatrix}6 & -4\\1 & 1\end{pmatrix}\begin{pmatrix}\lambda_1\\\lambda_2\end{pmatrix}\Leftrightarrow\begin{pmatrix}6-s & -4\\1 & 1-s\end{pmatrix}\begin{pmatrix}\lambda_1\\\lambda_2\end{pmatrix}=0$

Caract. equat. $s^2-7s+10=0$ $s=5$, $s=2$

$s=5$ $\left(\begin{array}{cc|c}1 & -4 & 0\\1 & -4 & 0\end{array}\right)$ $P_1\begin{bmatrix}4\\1\end{bmatrix}$ | $s=2$ $\left(\begin{array}{cc|c}4 & -4 & 0\\1 & -1 & 0\end{array}\right)$ $P_2\begin{bmatrix}1\\1\end{bmatrix}$

4.8.5) The best is to show that if there are two linear independent eigenvectors to a solution s to then $A=sI$. $P_1=v_1$ and $P_2=v_2$ such eigenvectors. Consider then the base points of the projectivity with matrix A then $s_1\begin{pmatrix}1\\0\end{pmatrix}=A\begin{pmatrix}1\\0\end{pmatrix}$ and $s_1\begin{pmatrix}0\\1\end{pmatrix}=A\begin{pmatrix}0\\1\end{pmatrix}$

and $A=\begin{pmatrix}s_1 & 0\\0 & s_1\end{pmatrix}$.

4.8.6) Prove that a projectivity that interchanges a pair of distinct elements is an involution: T

$T(P)=Q$ and $T(Q)=P$ $\Big\{$ use P and Q as base points

$T^2(P)=T(Q)=P$

$T^2(Q)=T(P)=Q$

$T^2(P+Q)=T^2(P)+T^2(Q)=P+Q$ $\;\Rightarrow T^2=Id$

4.9.1) Determine $R(A,B,C,D)$ and $R(C,A,B,D)$ of $A(1,1)$, $B(3,2)$, $C(1,0)$ and $D(-1,2)$

$$R(A,B,C,D) = \frac{\begin{vmatrix}1&1\\0&1\end{vmatrix} / \begin{vmatrix}1&3\\0&2\end{vmatrix}}{\begin{vmatrix}-1&1\\2&1\end{vmatrix} / \begin{vmatrix}-1&3\\2&2\end{vmatrix}} = \frac{1/2}{3/8} = \frac{4}{3}$$

$$R(C,A,B,D) = \frac{\begin{vmatrix}3&1\\2&0\end{vmatrix} / \begin{vmatrix}3&1\\2&1\end{vmatrix}}{\begin{vmatrix}-1&1\\2&0\end{vmatrix} / \begin{vmatrix}-1&1\\2&1\end{vmatrix}} = \frac{-2}{2/3} = -3$$

4.9.2 Find the coordinates of D such that $A(3,1,2)$, $B(1,0,-1)$, $C(1,1,4)$ and $R(A,B,C,D) = -2/3$

First of all, the line they belong to $\left(\begin{vmatrix}x_1&1&1\\x_2&0&1\\x_3&-1&4\end{vmatrix} = 0\right)$ has equation $x_1 - 5x_2 + x_3 = 0$

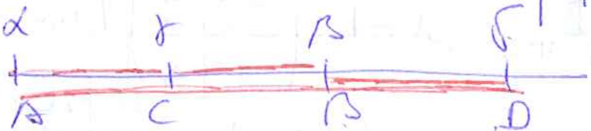
Nor $X(1,0,0)$ or $Z(0,0,1)$ belong to the line

We can use
$$\frac{\begin{vmatrix}1&3\\1&1\end{vmatrix} / \begin{vmatrix}1&1\\1&0\end{vmatrix}}{\begin{vmatrix}d_1&3\\d_2&1\end{vmatrix} / \begin{vmatrix}d_1&1\\d_2&0\end{vmatrix}} = -2/3$$
and also
$$\frac{\begin{vmatrix}1&0\\4&2\end{vmatrix} / \begin{vmatrix}1&0\\4&-1\end{vmatrix}}{\begin{vmatrix}d_2&1\\d_3&2\end{vmatrix} / \begin{vmatrix}d_2&0\\d_3&-1\end{vmatrix}} = -2/3$$
So
$$\begin{cases}\dfrac{(-d_2)(-2)}{(-1)(d_1-3d_2)} = -2/3\\[2ex] \dfrac{(-2)(-d_2)}{(-1)(2d_2-d_3)} = -2/3\end{cases}$$

$$\begin{cases}6d_2 = 2(d_1-3d_2) \Rightarrow d_1 = 6d_2\\ 6d_2 = 2(2d_2-d_3) \quad d_3 = -d_2\end{cases} \Rightarrow D\begin{bmatrix}-6\\-1\\1\end{bmatrix}$$

Obs that $D\begin{bmatrix}-6\\-1\\1\end{bmatrix}$ belongs to $m[1,-5,1]$.

4.9.3
$$R(A,B,C,D) = \frac{\begin{vmatrix}\gamma&\alpha\\1&1\end{vmatrix} / \begin{vmatrix}\gamma&\beta\\1&1\end{vmatrix}}{\begin{vmatrix}\delta&\alpha\\1&1\end{vmatrix} / \begin{vmatrix}\delta&\beta\\1&1\end{vmatrix}} = \frac{(\gamma-\alpha)/(\gamma-\beta)}{(\delta-\alpha)/(\delta-\beta)}$$



4.9.4
$R(A,B,C,D)$
$A(1,0)$, $B(0,1)$ $C(1,1)$, $D(r,1)$
$$= \frac{\begin{vmatrix}1&1\\1&0\end{vmatrix} / \begin{vmatrix}1&0\\1&1\end{vmatrix}}{\begin{vmatrix}r&1\\1&0\end{vmatrix} / \begin{vmatrix}r&0\\1&1\end{vmatrix}} = \frac{(r)(-1)}{(1)(-1)} = r$$

4.10.3 | Given the collineation with matrix

$$\begin{pmatrix} 2 & 0 & 1 \\ 0 & 2 & 3 \\ 0 & 0 & 1 \end{pmatrix} = A$$

a) Give the images of $P(1,2,3)$ and $Q(-1,0,1)$

$$s_1\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 2 & 0 & 1 \\ 0 & 2 & 3 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \quad P'\begin{pmatrix} 5/3 \\ 13/3 \\ 7 \end{pmatrix},\ Q'(-1, 3, 1)$$

b) $\begin{vmatrix} x_1 & 5 & -1 \\ x_2 & 13 & 3 \\ x_3 & 3 & 1 \end{vmatrix} = 0 \iff 2x_2 - x_1 + 7x_3 = 0 \Rightarrow l'[1, -2, 7]$

c) $PQ: \begin{vmatrix} x_1 & 1 & -1 \\ x_2 & 2 & 0 \\ x_3 & 3 & 1 \end{vmatrix} = 0 \iff x_1 - 2x_2 + x_3 = 0 \quad l[1,-2,1]$

then $s[2,-1,1] = [1,-2,1]\begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -3 \\ 0 & 0 & 2 \end{pmatrix} = [1,-2,7]$

4.10.6

$$s_1\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = A^{-1}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix};\ s_2\begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} = A^{-1}\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix};\ s_3\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} = A^{-1}\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} b_{11} & 2b_{12} & 0 \\ 0 & 0 & b_{23} \\ b_{11} & b_{12} & b_{23} \end{pmatrix}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \qquad \begin{matrix} 0 = b_{11} + b_{12}\cdot 2 \\ 2 = b_{23} \\ -1 = b_{11} + b_{12} \end{matrix} \quad \left| \begin{matrix} 1 = b_{12} \\ -2 = b_{11} \end{matrix} \right.$$

$$sA = s\begin{pmatrix} 1/2 & 0 & -1 \\ 1 & 1 & -1 \\ 0 & 1/2 & 0 \end{pmatrix}$$

4.10.8 Find the invariant points and lines of the collineations (a) $\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & a \end{pmatrix}$, $a \neq 0,1$; $\begin{vmatrix} 1-\lambda & 1 & \\ & 1-\lambda & \\ & & a-\lambda \end{vmatrix} = 0 \quad (1-\lambda)^2(a-\lambda)$

$\lambda = a \quad P_1\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix},\ \lambda = 1 \quad P_2\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$; the line between them is invariante

the coordinates to the line are $[0,1,0]$ and also $[0,0,1]$

(b) $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \quad \begin{vmatrix} 1-\lambda & 0 & 0 \\ 0 & 1-\lambda & 0 \\ 0 & 1 & 1-\lambda \end{vmatrix} = 0 \quad (1-\lambda)^3 = 0 : \lambda = 1 : \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$ line of invariant points

$m[0,1,0]$ For lines $\begin{vmatrix} 1-\lambda & 0 & 0 \\ 0 & 1-\lambda & 1 \\ 0 & 0 & 1-\lambda \end{vmatrix} = 0 ;\ \lambda = 1 ;\ \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \Rightarrow C\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

observe that $C \in m$, an elation

c) $\begin{pmatrix} a & 1 & 0\\ 0 & a & 1\\ 0 & 0 & a\end{pmatrix}$, $a\neq 0$ $\begin{vmatrix} a-\lambda & 1 & 0\\ 0 & a-\lambda & 1\\ 0 & 0 & a-\lambda\end{vmatrix}=0$; $(a-\lambda)^3=0$ $\cdot 3$

$\lambda = a$: $\left(\begin{array}{ccc|c} 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 0\end{array}\right) \to B\begin{bmatrix}0\\0\\1\end{bmatrix}$

m

For lines we controll the eigenvectors of A^t

$\begin{vmatrix} a-\lambda & 0 & 0\\ 1 & a-\lambda & 0\\ 0 & 1 & a-\lambda\end{vmatrix}=0$; $(a-\lambda)^3=0$; $\lambda=a$ $\left(\begin{array}{ccc|c} 0 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 0\end{array}\right)$

$m[0,0,1]$

4.10.12 Show that the collineation with the following matrix is a homology. Is it a harmonic homology?

$A=\begin{pmatrix} 1 & 0 & 0\\ 0 & 1 & 0\\ 2 & 0 & -1\end{pmatrix}$; $\begin{vmatrix} 1-\lambda & 0 & 0\\ 0 & 1-\lambda & 0\\ 2 & 0 & -1-\lambda\end{vmatrix}=0$ $(1+\lambda)(1-\lambda)^2=0$

$\lambda=1$ $\left(\begin{array}{ccc|c} 2 & 0 & -2 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{array}\right)$ $x_1=x_3$ line of invariant pts → perspective ~~homology~~

$m[1,0,-1]$

$\lambda=-1$ $\left(\begin{array}{ccc|c} 2 & 0 & 0 & 0\\ 0 & 2 & 0 & 0\\ 2 & 0 & 0 & 0\end{array}\right)$ $C\begin{bmatrix}0\\0\\1\end{bmatrix}$

lines: Eigenvectors to A^t $\begin{vmatrix} 1-\lambda & 0 & 2\\ 0 & 1-\lambda & 0\\ 0 & 0 & -1-\lambda\end{vmatrix}=0$; $(1+\lambda)(1-\lambda)^2=0$

$\lambda=1$ $\left(\begin{array}{ccc|c} 0 & 0 & 2 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & -2 & 0\end{array}\right)$ $x_3=0$ point $C\begin{bmatrix}0\\0\\1\end{bmatrix}$ of invariant lines

and $C\notin m$ → So, homology. Take $X\begin{bmatrix}1\\0\\0\end{bmatrix}$, $X\notin m$, $X\neq C$

$Q=CX\cdot m$; $\begin{vmatrix} n_1 & n_2 & n_3\\ 0 & 1 & 0\end{vmatrix}$ $n_1+n_3=0$ $Q\begin{bmatrix}1\\0\\-1\end{bmatrix}$

Calculate $R(C,Q,X,X')$ where $X'=s\begin{pmatrix} 1 & 0 & 0\\ 0 & 1 & 0\\ 2 & 0 & -1\end{pmatrix}\begin{pmatrix}1\\0\\0\end{pmatrix}=\begin{bmatrix}1\\0\\2\end{bmatrix}$

$Y\begin{bmatrix}0\\1\\0\end{bmatrix}\notin CX$ so

$$R(C,Q,X,X')=\frac{\begin{vmatrix}1 & 0\\ 0 & 1\end{vmatrix} / \begin{vmatrix}1 & 1\\ 0 & -1\end{vmatrix}}{\begin{vmatrix}1 & 0\\ 2 & 1\end{vmatrix} / \begin{vmatrix}1 & 1\\ 2 & -1\end{vmatrix}}=\frac{1/-1}{1/-3}=3\neq -1$$

not harmonic