Lösningar till Uppgifter 4.11

1) Given polarity with matrix $\begin{pmatrix} 2 & 0 & -1 \\ 0 & 1 & 1 \\ -\phi & 1 & 0 \end{pmatrix}$

(a) Find the equations of set of self-conjugate points and self-conjugate lines

(b) Find the pole of $l[1,1,1]$, c) find a point conjugate to $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

a) $C^{-1} = \begin{pmatrix} 1 & 1 & -1 \\ 1 & 1 & 2 \\ -1 & 2 & -2 \end{pmatrix}$:

Point-conic $(x_1, x_2, x_3)\, C \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$

$: \{ 2x_1^2 + x_2^2 - 2x_1x_3 + 2x_2x_3 = 0$

Line-conic $[m_1, m_2, m_3]\, C^{-1} \begin{bmatrix} m_1 \\ m_2 \\ m_3 \end{bmatrix} = 0$

$\{ m_1^2 + m_2^2 - 2m_3^2 + 2m_1m_2 - 2m_1m_3 + 4m_2m_3 = 0$

b) Pole of $l[1,1,1]$: lösning av systemet

$$\left(\begin{array}{ccc|c} 2 & 0 & -1 & 1 \\ 0 & 1 & 1 & 1 \\ -1 & 1 & 0 & 1 \end{array}\right) \Rightarrow \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}$$

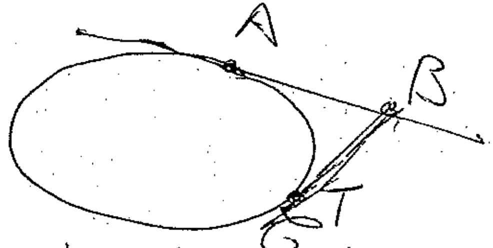
c) Polar to $P\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ is $\begin{pmatrix} 2 & 0 & -1 \\ 0 & 1 & 1 \\ -1 & 1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = [1, 2, 0]^t$

$x_1 - 2x_2 = 0$

Any point conjugate to $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ is $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ or $\begin{pmatrix} 2 \\ 1 \\ 6 \end{pmatrix}$ $6\,4\,1$

$\boxed{4.11.5}$ Prove that if A is a point of $\mathcal{C}$ and $B \neq A$ is a second point on the polar of A, then the polar of B contains exactly two points of $\mathcal{C}$.

To show this look at



B is exterior to $\mathcal{C}$, so there is a second tangent to $\mathcal{C}$ from B. with tangency point T.

$B \in$ polar A $\wedge$ $B \in$ polar T, so <u>polar $B = AT$</u>

$\longmapsto$

$\boxed{4.11.8}$ Given the conic with matrix

$\begin{pmatrix} 1 & -2 & -1 \\ -2 & 2 & -1 \\ -1 & -1 & 5 \end{pmatrix}$ find
- a) the tangent at $(1,1,1)$
- b) the polar of $\begin{pmatrix} 3 \\ 1 \\ 5 \end{pmatrix}$
- c) the tangents from the point $\begin{pmatrix} 1 \\ -3 \\ 0 \end{pmatrix}$

a) m = polar of $P\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ = tangent at P; we have $m^t = \begin{pmatrix} 1 & -2 & -1 \\ -2 & 2 & -1 \\ -1 & -1 & 5 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = [-2, -1, 3$

$$\boxed{2x_1 + x_2 + 3x_3 = 0}$$

b) Again q = polar of $Q\begin{pmatrix} 3 \\ 1 \\ 5 \end{pmatrix}$; $\begin{pmatrix} 1 & -2 & -1 \\ -2 & 2 & -1 \\ -1 & -1 & 5 \end{pmatrix}\begin{pmatrix} 3 \\ 1 \\ 5 \end{pmatrix} = [-4, -9, 1]$

$$\boxed{4x_1 + 9x_2 - x_3 = 0}$$

c) First, polar to $R\begin{pmatrix} 1 \\ -3 \\ 0 \end{pmatrix}$; $\begin{pmatrix} 1 & -2 & -1 \\ -2 & 2 & -1 \\ -1 & -1 & 5 \end{pmatrix}\begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix} = [5, -6, 1]^t$

$$5x_1 - 6x_2 + x_3 = 0$$

Secondly; points of intersection with $\mathcal{C}$

$$\begin{cases} x_1^2 + 2x_2^2 + 5x_3^2 - 4x_1x_2 - 2x_1x_3 - 2x_2x_3 = 0 \\ x_3 = -5x_1 + 6x_2 \end{cases} \qquad \begin{cases} x_2 = 1 \\ x_1 = \dfrac{85 \pm \sqrt{48831}}{68} \end{cases} \quad \begin{matrix} Q_1 \\ Q_2 \end{matrix}$$

Thirdly: Determine lines through RQ_1 och RQ_2.

4.11.15 Find the matrix of a collineation T such that the conic $\mathcal{C}: ax_1^2+bx_2^2+cx_3^2$, $a<0$, $b>0$, $c>0$, transforms to the conic with equation $a'x_1^2+b'x_2^2+c'x_3^2=0$ with $a'>0$, $b'>0$, $c'<0$.

We see that the rôles of x_1 and x_3 are changed, so take a collineation that takes $X\begin{pmatrix}1\\0\\0\end{pmatrix}$ to $Z\begin{pmatrix}0\\0\\1\end{pmatrix}$, $Y\begin{pmatrix}0\\1\\0\end{pmatrix}$ to $Y\begin{pmatrix}0\\1\\0\end{pmatrix}$, $Z\begin{pmatrix}0\\0\\1\end{pmatrix}$ to $X\begin{pmatrix}1\\0\\0\end{pmatrix}$ and $U\begin{pmatrix}1\\1\\1\end{pmatrix}$ to $U\begin{pmatrix}1\\1\\1\end{pmatrix}$. This collineation has matrix

As usual: $\begin{pmatrix}0\\0\\s_1\end{pmatrix}=A\begin{pmatrix}1\\0\\0\end{pmatrix}$; $\begin{pmatrix}0\\s_2\\0\end{pmatrix}=A\begin{pmatrix}0\\1\\0\end{pmatrix}$, $\begin{pmatrix}s_3\\0\\0\end{pmatrix}=A\begin{pmatrix}0\\0\\1\end{pmatrix}$

and $\begin{pmatrix}s_4\\s_4\\s_4\end{pmatrix}=\begin{pmatrix}0&0&s_3\\0&s_2&0\\s_1&0&0\end{pmatrix}\begin{pmatrix}1\\1\\1\end{pmatrix} \Rightarrow s_3=s_2=s_1=1$ $\boxed{A=\begin{pmatrix}0&0&1\\0&1&0\\1&0&0\end{pmatrix}}$

$s_4=1$

4.11.18 Hint to part a) The correlation T takes colinear points to concurrent lines and concurrent lines to colinear points