Hand-in Exercises 2, TATA49 Geometry with Applications

Maple or Matlab may be used in the calculations.

The exercises are customized by your birthdays as in the previous sheet of exercises.

Exercise 1 *Do your computer aided portrait with instances of a rhomb with vertices* $O(0,0)$, $R_1(q_1+r_1, 5+r_2)$, $R_2(q_1+r_1, -5-r_2)$, $R_3(2q_1+2r_1, 0)$ *. First of all transform the rhomb such that the side has length 50. The mouth has ratios 1:15 and 1:100 to the face and lies centered at the diagonal of the rhomb (the height of the face) dividing the height in ratio 1:9 from the bottom. The two eyes lie (also symmetric) on a hight 3:4 to the rhomb's hight; they have ratio 1:20 to the face. Which affine transformations do you have to do? Give their matrices.*

Exercise 2 *Determine the images under the stereographic projection* $\varphi : \widehat{\mathbb{C}} \to \mathbb{S}^2$ *of all the circles in* $\mathbb{C}$ *with equations* $x^2 + y^2 - 2ax - 2ay = 0$ *, with* $a \neq 0$ *a parameter.*

Exercise 3 *Use spherical coordinates* $\mathbf{x}(\lambda, \varphi) = (\cos(\lambda)\cos(\varphi), \sin(\lambda)\cos(\varphi), \sin(\varphi))$ *, where* λ *is the longitude and* φ *is the latitude to provide a parametrization of the circles obtained in the previous exercise using as parameter the longitude* λ *.*

Exercise 4 *Determine the images under the inverse of stereographic projection* $\phi : \mathbb{S}^2 \to \widehat{\mathbb{C}}$ *of all the circles in* $\mathbb{S}^2$ *determined by the planes* $x_2 - x_3 = c$ *, with* $-\sqrt{2} < c < \sqrt{2}$ *a parameter.*

Exercise 5 *Consider the circles on* $\mathbb{S}^2$ *in the previous exercise. What can you say on the circle corresponding to the parameter* $c = -1$ *? And on its image by* ϕ *? Determine the circles(s) in the family that are great circles and calculate its(their) distance to the North Pole* N *.*

Exercise 6 ***a)*** *Calculate the angles in the spherical triangle with vertices A (longitude* $(r_1{+}3)(p_1{+}p_2)$ *deg, latitude 0 deg), B (longitude 0 deg, latitude* $q_2q_1{+}30$ *deg) and C (longitude* $(r_1{-}2)p_2$ *deg, latitude 60 deg). Observation* $(r_1+3)(p_1+p_2)$ *is a number with three digits:* p_1+p_2 *in the units and* r_1+3 *in the tens and hundreds. In the same way* q_2q_1 *is a number with two digits:* q_1 *in the units and* q_2 *in the tens.*

b) *Calculate the area of the triangle if we consider that the triangle is on Earth (Radius of Earth is 6371 km).*

Exercise 7 ***a)*** *Determine the great circles that support the sides of the spherical triangle in exercise 5 and their images under the inverse of the stereographic projection.*

b) *Determine the sizes of the edges of the spherical triangle (on Earth) in exercise 4 (Radius of Earth is 6371 km).*

Exercise 8 *Consider* $\mathbb{R}^4$ *with coordinates* $\mathbf{v} = (v_1, v_2, v_3, v_0)$ *(Observe that the real part of the quaternion is the fourth coordinate). Given the quaternion* $q_0 = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k} + a_0 = ((a_1, a_2, a_3), a_0)$ *, show that the multiplication with* q_0 *to the left is a linear transformation in* $\mathbb{R}^4$ *with matrix*

$$\begin{pmatrix} a_0 & -a_3 & a_2 & a_1 \\ a_3 & a_0 & -a_1 & a_2 \\ -a_2 & a_1 & a_0 & a_3 \\ -a_1 & -a_2 & -a_3 & a_0 \end{pmatrix}.$$

Calculate the matrix for the multiplication with q_0 *on the right.*

Exercise 9

1. *Use Exercise 8 to determine the matrix of the rotation in* $\mathbb{A}^3$ *with quaternion* $q_0 = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k} + a_0$, $|q_0| = 1$ *.*
2. *Use Exercise 8 to determine the matrix of the reflection in* $\mathbb{A}^3$ *on the plane with equation* $6x_1 - 2x_2 - 3x_3 = 0$ *. Observe that the last row and column of the matrix equal* $(0,0,0,1)$ *.*

Exercise 10 *Consider an animation of an object with start orientation (time* $t = 0$ *) given by the quaternion* $q_s = (r_1, r_2, p_1+p_2, q_1+q_2)/l$ *where* $l = \sqrt{(q_1+q_2)^2 + (p_1+p_2)^2 + (r_2)^2 + (r_1)^2}$ *and final orientation (time* $t = 1$ *) given by the quaternion* $q_f = (4/7, (4/7, 1/7, 4/7))$ *. Give the orientations at times* $t = j/10,\ j = 0, 1, 2, \ldots, 10$ *.*
Use Exercise 9.1 to give the matrices of the rotations corresponding to the orientations for times $t_1 = (p_1 + p_2 + q_1)/10$ *and* $t_1 = (2p_1 + 2p_2)/10$

Exercise 11 *Identify and describe completely the isometry of* $\mathbb{A}^3$ *with matrix:*
$$\begin{pmatrix} 7/9 & 4/9 & 4/9 & -1 \\ 4/9 & 1/9 & -8/9 & 1 \\ 4/9 & -8/9 & 1/9 & 2 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$
Start by determining its fixed points, if existing.

Exercise 12 *Let* q *be a quaternion. Show that* q *is a pure quaternion iff* q^2 *is a non-positive real number.*
Given two quaternions p, q *. Show that* $pq = qp$ *iff their vectorial parts are parallel. Is the statement true?*

Exercise 13 *Consider a computer animation consisting only of translations. First translation* T_0 *has vector* $\mathbf{v}_0$ *and last translation* T_1 *has vector* $\mathbf{v}_1$ *. Show that any intermediate translation* $T(t)$ *,* $0 \le t \le 1$ *, in the animation can be written* $T(t) = (1-t)T_0 + tT_1$ *. Give the matrix of any intermediate translation.*

Exercise 14 *Describe the symmetries of the friezes below:*