Some examples and commands useful in Linear Geometry

The first thing to be done in Maple is to loaded the packages with routines: LinearAlgbra, plots, stats. There is a package Geometry, but it **should not** be used.

The routines from linear algebra used by us are *CharacteristicMatrix*, *CrossProduct*, *Determinant*, *DotProduct*, *Eigenvalues*, *Eigenvectors*, *GaussianElimination*, *JordanBlockMatrix*, *JordanForm*, *LinearSolve*, *MatrixInverse*, *MatrixMatrixMultiply*, *MatrixNorm*, *MatrixPower*, *MatrixScalarMultiply*, *MatrixVectorMultiply*, *MinimalPolynomial*, *Minor*, *Modular*, *Multiply*, *NullSpace*, *Transpose*,

```
with(plots);
```

[*animate, animate3d, animatecurve, arrow, changecoords, complexplot, complexplot3d, conformal, conformal3d, contourplot, contourplot3d, coordplot, coordplot3d, densityplot, display, dualaxisplot, fieldplot, fieldplot3d, gradplot, gradplot3d, implicitplot, implicitplot3d, inequal, interactive, interactiveparams, intersectplot, listcontplot, listcontplot3d, listdensityplot, listplot, listplot3d, loglogplot, logplot, matrixplot, multiple, odeplot, pareto, plotcompare, pointplot, pointplot3d, polarplot, polygonplot, polygonplot3d, polyhedra_supported, polyhedraplot, rootlocus, semilogplot, setcolors, setoptions, setoptions3d, shadebetween, spacecurve, sparsematrixplot, surfdata, textplot, textplot3d, tubeplot*] **(1)**

```
with(LinearAlgebra); with(SolveTools);
```

[*&x, Add, Adjoint, BackwardSubstitute, BandMatrix, Basis, BezoutMatrix, BidiagonalForm, BilinearForm, CARE, CharacteristicMatrix, CharacteristicPolynomial, Column, ColumnDimension, ColumnOperation, ColumnSpace, CompanionMatrix, CompressedSparseForm, ConditionNumber, ConstantMatrix, ConstantVector, Copy, CreatePermutation, CrossProduct, DARE, DeleteColumn, DeleteRow, Determinant, Diagonal, DiagonalMatrix, Dimension, Dimensions, DotProduct, EigenConditionNumbers, Eigenvalues, Eigenvectors, Equal, ForwardSubstitute, FrobeniusForm, FromCompressedSparseForm, FromSplitForm, GaussianElimination, GenerateEquations, GenerateMatrix, Generic, GetResultDataType, GetResultShape, GivensRotationMatrix, GramSchmidt, HankelMatrix, HermiteForm, HermitianTranspose, HessenbergForm, HilbertMatrix, HouseholderMatrix, IdentityMatrix, IntersectionBasis, IsDefinite, IsOrthogonal, IsSimilar, IsUnitary, JordanBlockMatrix, JordanForm, KroneckerProduct, LA_Main, LUDecomposition, LeastSquares, LinearSolve, LyapunovSolve, Map, Map2, MatrixAdd, MatrixExponential, MatrixFunction, MatrixInverse,* *MatrixMatrixMultiply*, *MatrixNorm*, *MatrixPower*, *MatrixScalarMultiply*, *MatrixVectorMultiply*, *MinimalPolynomial*, *Minor*, *Modular*, *Multiply*, *NoUserValue*, *Norm*, *Normalize*, *NullSpace*, *OuterProductMatrix*, *Permanent*, *Pivot*, *PopovForm*, *ProjectionMatrix*, *QRDecomposition*, *RandomMatrix*, *RandomVector*, *Rank*, *RationalCanonicalForm*, *ReducedRowEchelonForm*, *Row*, *RowDimension*, *RowOperation*, *RowSpace*, *ScalarMatrix*, *ScalarMultiply*, *ScalarVector*, *SchurForm*, *SingularValues*, *SmithForm*, *SplitForm*, *StronglyConnectedBlocks*, *SubMatrix*, *SubVector*, *SumBasis*, *SylvesterMatrix*, *SylvesterSolve*, *ToeplitzMatrix*, *Trace*, *Transpose*, *TridiagonalForm*, *UnitVector*, *VandermondeMatrix*, *VectorAdd*, *VectorAngle*, *VectorMatrixMultiply*, *VectorNorm*, *VectorScalarMultiply*, *ZeroMatrix*, *ZeroVector*, *Zip*]

[*AbstractRootOfSolution*, *Basis*, *CancelInverses*, *Combine*, *Complexity*, *Engine*, *GreaterComplexity*, *Identity*, *Inequality*, *Linear*, *Parametric*, *Polynomial*, *PolynomialSystem*, *RationalCoefficients*, *SemiAlgebraic*, *SortByComplexity*] **(2)**

1) In this example we multiply the matrices A of isometries (parametrised by the angle of rotation and the translation vector) with the points in a segment (parametrised by a parametre x). The result is an annulus, that we plot.

$$A := Matrix\left(\left[\left[\cos\left(\frac{2\cdot \mathrm{Pi}\cdot t}{10}\right), -\sin\left(\frac{2\cdot\pi\cdot t}{10}\right), 5-s\right], \left[\sin\left(\frac{2\cdot\pi\cdot t}{10}\right), \cos\left(\frac{2\cdot\pi\cdot t}{10}\right), 3+s\right], [0, 0, 1]\right]\right);$$

$$A := \begin{bmatrix} \cos\left(\frac{\pi\ t}{5}\right) & -\sin\left(\frac{\pi\ t}{5}\right) & 5-s \\ \sin\left(\frac{\pi\ t}{5}\right) & \cos\left(\frac{\pi\ t}{5}\right) & 3+s \\ 0 & 0 & 1 \end{bmatrix} \quad \mathbf{(3)}$$

$$\mathrm{v} := \mathrm{Vector}([[3-\mathrm{x}], [5-x], [1]]);$$

$$v := \begin{bmatrix} 3-x \\ 5-x \\ 1 \end{bmatrix} \quad \mathbf{(4)}$$

$$\mathrm{w} := \mathrm{Vector}(3, 1, [3, 5, 1]);$$

$$w := \begin{bmatrix} 3 \\ 5 \\ 1 \end{bmatrix} \tag{5}$$

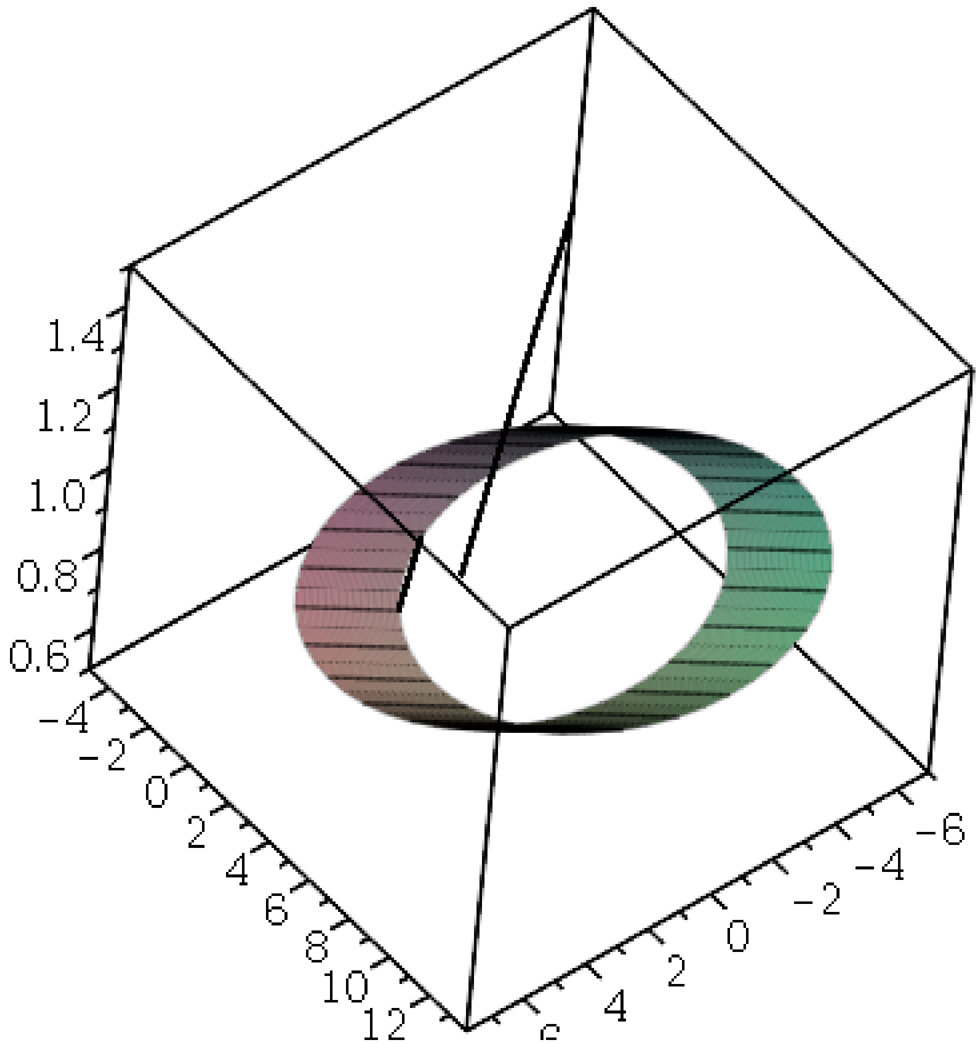
$R := MatrixVectorMultiply(A, w);$

$$R := \begin{bmatrix} 3\cos\left(\frac{\pi\, t}{5}\right) - 5\sin\left(\frac{\pi\, t}{5}\right) + 5 - s \\ 3\sin\left(\frac{\pi\, t}{5}\right) + 5\cos\left(\frac{\pi\, t}{5}\right) + 3 + s \\ 1 \end{bmatrix} \tag{6}$$

$S := MatrixVectorMultiply(A, v);$

$$S := \begin{bmatrix} \cos\left(\frac{\pi\, t}{5}\right)(3 - x) - \sin\left(\frac{\pi\, t}{5}\right)(5 - x) + 5 - s \\ \sin\left(\frac{\pi\, t}{5}\right)(3 - x) + \cos\left(\frac{\pi\, t}{5}\right)(5 - x) + 3 + s \\ 1 \end{bmatrix} \tag{7}$$

$plot3d\left(\left\{\left[3\cos\left(\frac{1}{5}\,\pi\, t\right) - 5\sin\left(\frac{1}{5}\,\pi\, t\right) + 5 - s, 3\sin\left(\frac{1}{5}\,\pi\, t\right) + 5\cos\left(\frac{1}{5}\,\pi\, t\right) + 3 + s, 1\right], [3 - t, 5 - t, 1], [s, s, 1]\right\}, t = 0..10, s = 3.00..5.00, axes = boxed\right);$



2) The following example is like the previous but with movements by rotations

$AA := matrix\left(\left[\left[\cos\left(\pi\cdot\frac{s}{100}\right), -\sin\left(\frac{\pi\cdot s}{100}\right), 0\right], \left[\sin\left(\pi\cdot\frac{s}{100}\right), \cos\left(\pi\cdot\frac{s}{100}\right), 0\right], [0, 0, 1]\right]\right);$

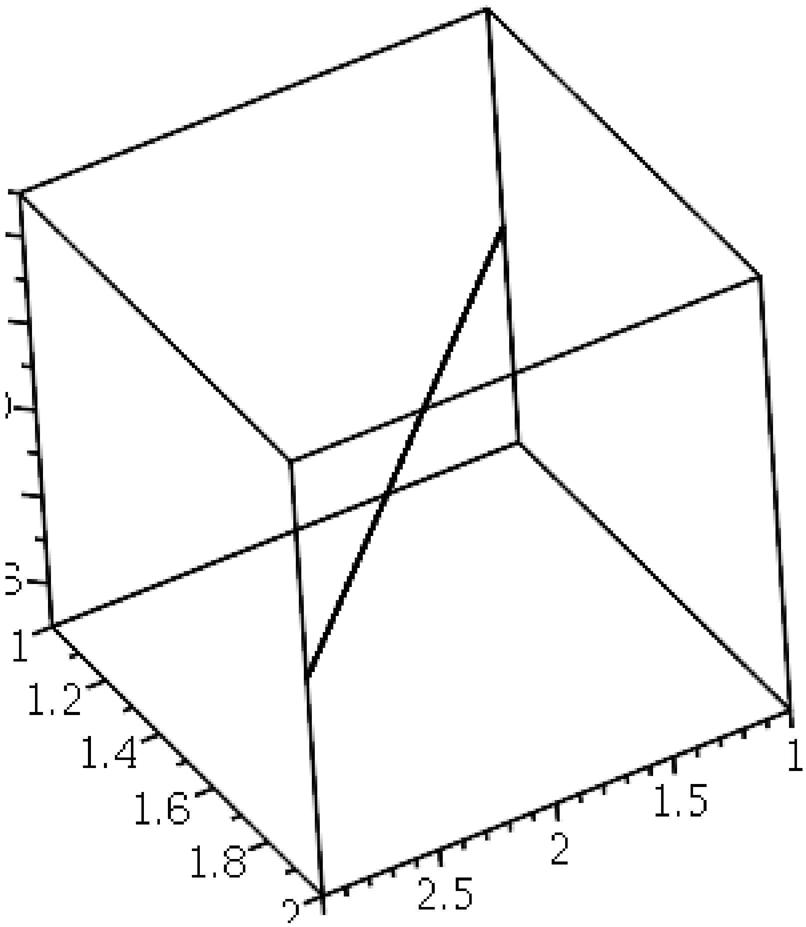
$$AA := \begin{bmatrix} \cos\left(\frac{\pi\, s}{100}\right) & -\sin\left(\frac{\pi\, s}{100}\right) & 0 \\ \sin\left(\frac{\pi\, s}{100}\right) & \cos\left(\frac{\pi\, s}{100}\right) & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \textbf{(8)}$$

$P1 := vector([1, 1, 1]) : P2 := vector([3, 2, 1]) :$
for m **from** 1 **to** 100 **do**

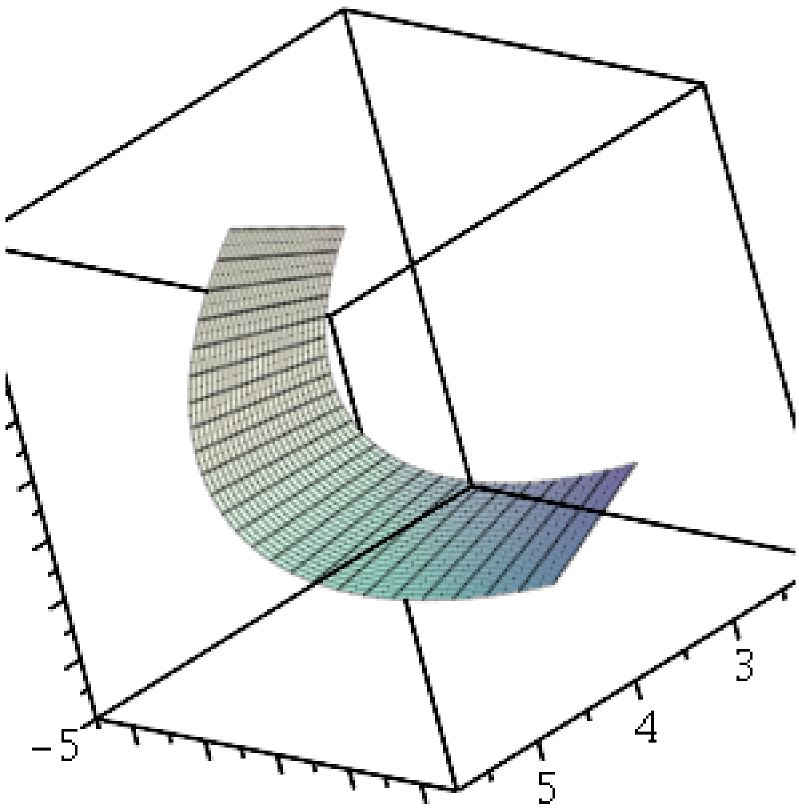
$P := vector\left(\left[\left(1-\frac{m}{100}\right)\cdot 1+\frac{m}{100}\cdot 3,\ \left(1-\frac{m}{100}\right)\cdot 1+\frac{m}{100}\cdot 2,\ 1\right]\right):$

od

$plot3d\left(\left[1+\frac{2}{100}\cdot t+s,\ 1+\frac{t}{100}\ +s,\ 1\right],\ t=0..100,\ s=0..0,\ axes=boxed\right);$



$plot3d\left(\left[\cos(s)\ \cdot(3-t)-\sin(s)\ \cdot(5-t),\ \sin(s)\ \cdot(3-t)+\cos(s)\ \cdot(5-t),\ 1\right],\ s=0\ ..\frac{\pi}{2},\ t=0..1,\ axes=boxed\right);$



3) This example produces three differents fractals by randomly choosing which similarity we apply to the first point P (the origin)

```
with(stats[random]), with(plots) :
Mat1 := linalg[matrix]([[0.5, 0.0], [0.0, 0.5]]);
Mat2 := linalg[matrix]([[0.5, 0.0], [0.0, 0.5]]);
Mat3 := linalg[matrix]([[0.5, 0.0], [0.0, 0.5]]);
Vector1 := linalg[vector]([0, 0]);
Vector2 := linalg[vector]([0, 1]) :
Vector3 := linalg[vector]([1, 1]) :
 Prob1 := 1 / 3 :
Prob2 := 1 / 3 :
Prob3 := 1 / 3 :
P := linalg[vector]([0, 0]) :
 for m from 1 to 1000 do
prob := uniform( ) :
if prob < Prob1 then P := evalm(Mat1&*P + Vector1)
elif prob < Prob1 + Prob2 then P := evalm(Mat2&*P + Vector2) else P := evalm(Mat3
    &*P + Vector3);
 fi:
 A[m, 1] := P[1] : A[m, 2] := P[2] :
```

od:
$C := matrix(1000, 2, (i, j) \rightarrow A[i, j])$:
$pointplot(C, scaling = constrained, axes = none, color = green)$;

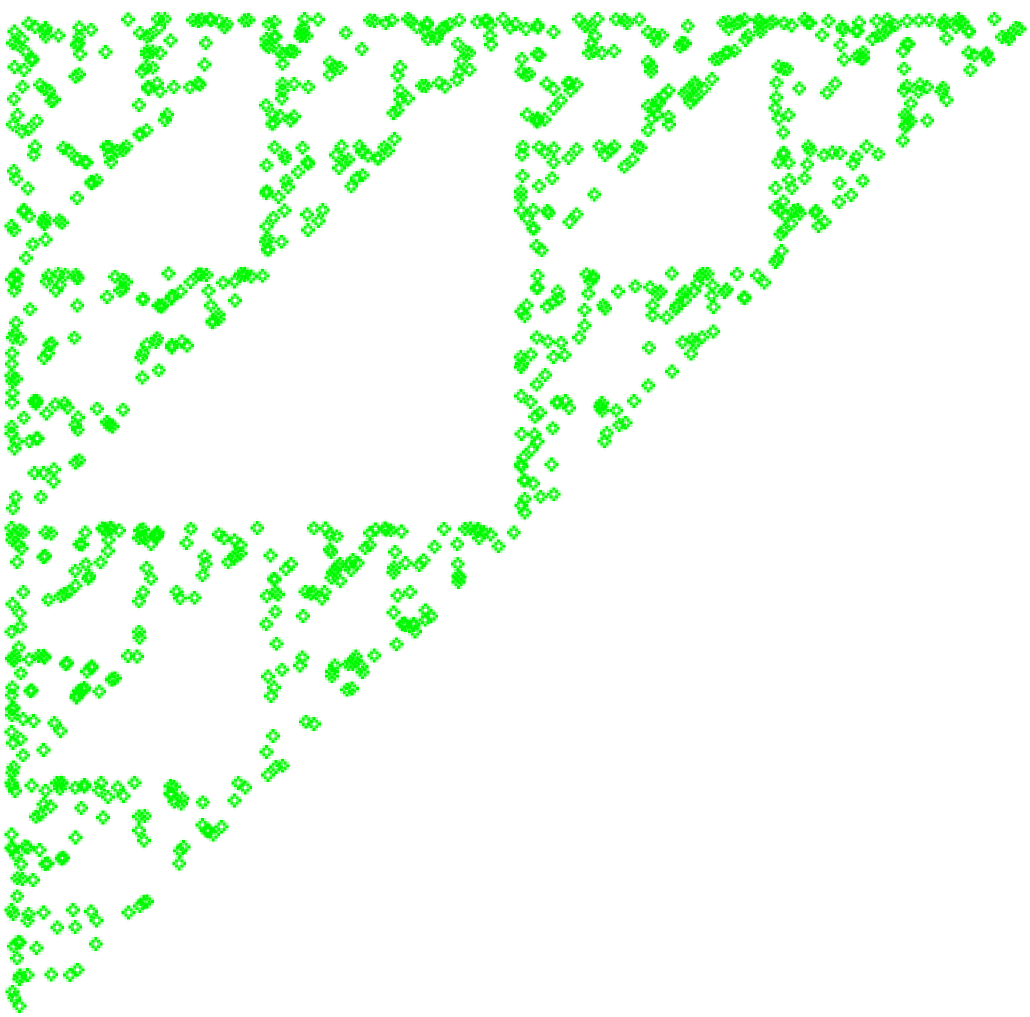
$$Mat1 := \begin{bmatrix} 0.5 & 0. \\ 0. & 0.5 \end{bmatrix}$$

$$Mat2 := \begin{bmatrix} 0.5 & 0. \\ 0. & 0.5 \end{bmatrix}$$

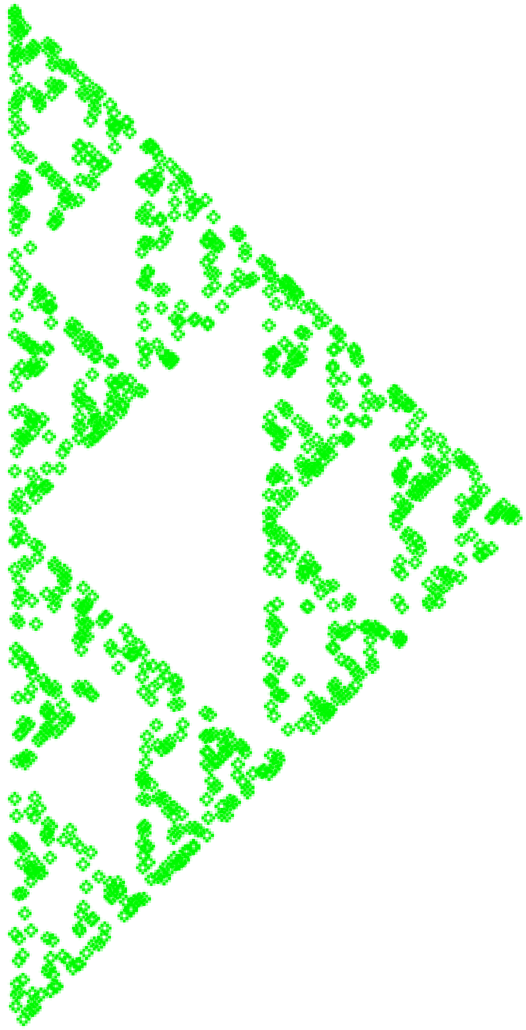
$$Mat3 := \begin{bmatrix} 0.5 & 0. \\ 0. & 0.5 \end{bmatrix}$$

$$Vector1 := \begin{bmatrix} 0 & 0 \end{bmatrix}$$

Error, Matrix index out of range
Error, (in unknown) Matrix index out of range

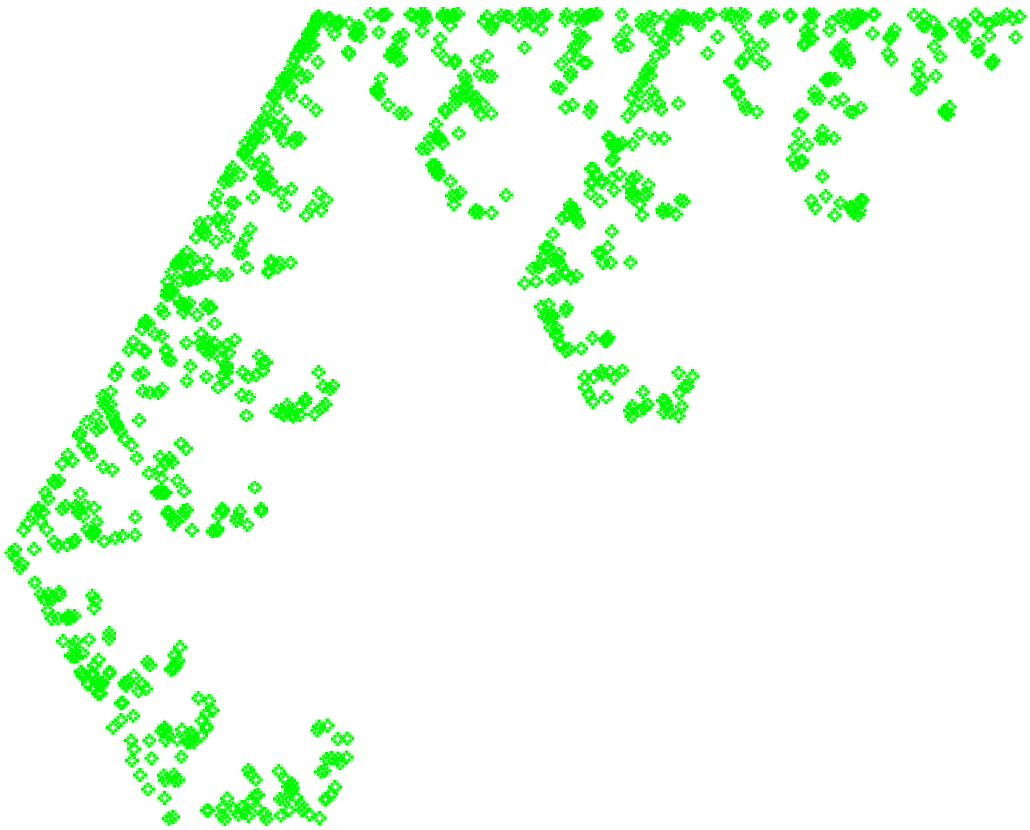


```
with(stats[random]), with(plots) :
Mat1 := linalg[matrix]([[0.5, 0.0], [0.0, 0.5]]) :
Mat2 := linalg[matrix]([[0.5, 0.0], [0.0, 0.5]]) :
Mat3 := linalg[matrix]([[0.5, 0.0], [0.0, 0.5]]) :
Vector1 := linalg[vector]([0, 0]) :
Vector2 := linalg[vector]([0, 1]) :
Vector3 := linalg[vector]([0.5, 0.5]) :
 Prob1 := 1 / 3 :
Prob2 := 1 / 3 :
Prob3 := 1 / 3 :
P := linalg[vector]([0, 0]) :
 for m from 1 to 1000 do
prob := uniform( ) :
if prob < Prob1 then P := evalm(Mat1&*P + Vector1)
elif prob < Prob1 + Prob2 then P := evalm(Mat2&*P + Vector2) else P := evalm(Mat3
    &*P + Vector3);
 fi:
 B[m, 1] := P[1] : B[m, 2] := P[2] :
 od:
 F := matrix(1000, 2, (i, j) → B[i, j]) :
 pointplot(F, scaling = constrained, axes = none, color = green);
```



```
with(stats[random]), with(plots) :
Mat1 := linalg[matrix]([[0.25,-0.43], [0.43, 0.25]]) :
Mat2 := linalg[matrix]([[0.5, 0.0], [0.0, 0.5]]) :
Mat3 := linalg[matrix]([[0.5, 0.0], [0.0, 0.5]]) :
Vector1 := linalg[vector]([0, 0]) :
Vector2 := linalg[vector]([0, 1]) :
Vector3 := linalg[vector]([1, 1]) :
 Prob1 := 1 / 3 :
Prob2 := 1 / 3 :
Prob3 := 1 / 3 :
P := linalg[vector]([0, 0]) :
 for m from 1 to 1000 do
prob := uniform( ) :
if prob < Prob1 then P := evalm(Mat1&*P + Vector1)
elif prob < Prob1 + Prob2 then P := evalm(Mat2&*P + Vector2) else P := evalm(Mat3
    &*P + Vector3);
 fi:
 A2[m, 1] := P[1] : A2[m, 2] := P[2] :
 od:
 C2 := matrix(1000, 2, (i, j) → A2[i, j]) :
```

pointplot(*C2*, *scaling* = *constrained*, *axes* = *none*, *color* = *green*);



4) Doing the same for a Moebius transformation,

$$Mat2 := Matrix\left(\left[\left[\frac{(3\cdot m-5)}{5\cdot m}, -\frac{(4\cdot m-5)}{5\ m}, 0\right], \left[\frac{(4\cdot m-5)}{5\ m}, \frac{(3\cdot m-5)}{5\cdot m}, 0\right], [0.0, 0.0, 1.0]\right]\right):$$

$P := vector([1, 0, 1]) : Q := vector([1, 1, 1]) :$
for m **from** 1 **to** 100 **do**
$P := evalm(Mat2 \&* P) : Q := evalm(Mat2 \&* Q) :$
$A[m, 1] := P[1] : A[m, 2] := P[2] : B[m, 1] := Q[1] : B[m, 2] := Q[2] :$
od:
$C := Matrix(100, 2, (i, j) \rightarrow A[i, j]) : E := Matrix(100, 2, (i, j) \rightarrow B[i, j]) :$
pointplot(*C*, *scaling* = *constrained*, *axes* = *none*, *color* = *green*); *pointplot*(*E*, *scaling* = *constrained*, *axes* = *none*, *color* = *green*);

5) This example provides the matirx of the configuration of a planar robot with two junctions (angles t1, t2), a rigid link of length l and periscopic hand of length s, with s∈ [0, s1]. The matrix is the product, in this order, A_{h2} the translation matrix with vector (s,0), A_{21} is the matrix of the rotation taking the system from link 1 to the hand, junction 2, followed by the translation with length the length of link 1, finally the matrix A_{10} is the matrix of the rotation with angle the angle at junction 1. The parameters are the angles t1, t2, the length l of the link and the parameter s of the distance between the hand and the junction 2.

Ah2 ≔ *Matrix*([[1, 0, *s*], [0, 1, 0], [0, 0, 1]]) :
A21 ≔ *Matrix*([[cos(*t2*),-sin(*t2*), *l*], [sin(*t2*), cos(*t2*), 0], [0, 0, 1]]) :
A10 ≔ *Matrix*([[cos(*t1*),-sin(*t1*), 0], [sin(*t1*), cos(*t1*), 0], [0, 0, 1]]) :
Conf ≔ *MatrixMatrixMultiply*(*A10*, *MatrixMatrixMultiply*(*A21*, *Ah2*));

$$Conf := [[\cos(t1)\cos(t2) - \sin(t1)\sin(t2), -\cos(t1)\sin(t2) - \sin(t1)\cos(t2), \cos(t1)(\cos(t2)\, s + l) - \sin(t1)\sin(t2)\, s], [\sin(t1)\cos(t2) + \cos(t1)\sin(t2), \cos(t1)\cos(t2) - \sin(t1)\sin(t2), \sin(t1)(\cos(t2)\, s + l) + \cos(t1)\sin(t2)\, s], [0, 0, 1]] \quad (9)$$

Observe that the configuration gives the total angle t1+t2 and the vector providing the coordinates of the hand. It gives the relation between the coordinates of a point as seen by the hand, Xh, and the coordinates of the same point as seen by the controller, X0: ConfXh =X0

Using the command **solve** we can calculate (here algebraically) the parameters t1, t2 and s (l is a constant of the system) for given Xh and X0. This is what the controller does to move the hand properly, giving the angles and the lenght of the periscope. This problem is much more complicated. **Observe that in the system is missing an equation: t1+t2= total angle. I have not writen the equation because I pick coordinates Xh(1/2, 1/2, 1) and X0(3,3,1) randomly and I write the three equations the probability the system to be inconsistent is high**.

$$solve\left(\left\{\frac{\cos(t1+t2)\cdot 1}{2}-\frac{\sin(t1+t2)\cdot 1}{2}+\cos(t1)\cdot(\cos(t2)\cdot s+2)-\sin(t1)\cdot\sin(t2)\cdot s=3, \frac{\sin(t1+t2)\cdot 1}{2}+\frac{\cos(t1+t2)\cdot 1}{2}+\sin(t1)\cdot(\cos(t2)\cdot s+l)+\cos(t1)\cdot\sin(t2)\cdot s=3\right\}, [t1, t2, s]\right);$$

$$\Big[\Big[t1=t1,\ t2=\arctan\Big(RootOf\big((4\sin(t1)^2 l^2-24\sin(t1)\,l-16\sin(t1)^2 -48\cos(t1)+88)\,\_Z^2+(-4\sin(t1)^2 l+8\sin(t1)^2+12\cos(t1) +12\sin(t1)-8)\,\_Z+4\sin(t1)^4 l^2-16\sin(t1)^4 l-24\cos(t1)\,l\sin(t1)^2 -4\sin(t1)^2 l^2-24\sin(t1)^3 l+16\sin(t1)^4+48\cos(t1)\sin(t1)^2 +16\sin(t1)^2 l+48\sin(t1)^3+72\cos(t1)\sin(t1)+24\sin(t1)\,l -16\sin(t1)^2-48\sin(t1)-35\big), \big((2\sin(t1)^2 l-4\sin(t1)^2-6\sin(t1) -6\cos(t1)+4)\,RootOf\big((4\sin(t1)^2 l^2-24\sin(t1)\,l-16\sin(t1)^2 -48\cos(t1)+88)\,\_Z^2+(-4\sin(t1)^2 l+8\sin(t1)^2+12\cos(t1) +12\sin(t1)-8)\,\_Z+4\sin(t1)^4 l^2-16\sin(t1)^4 l-24\cos(t1)\,l\sin(t1)^2 -4\sin(t1)^2 l^2-24\sin(t1)^3 l+16\sin(t1)^4+48\cos(t1)\sin(t1)^2 \tag{10}$$

$$+ 16 \sin(t1)^2 l + 48 \sin(t1)^3 + 72 \cos(t1) \sin(t1) + 24 \sin(t1) l$$

$$- 16 \sin(t1)^2 - 48 \sin(t1) - 35))\Big/ (2 \sin(t1) \cos(t1) l - 4 \cos(t1) \sin(t1)$$

$$+ 6 \sin(t1) - 6 \cos(t1))$$

$$- \frac{1}{2 \sin(t1) \cos(t1) l - 4 \cos(t1) \sin(t1) + 6 \sin(t1) - 6 \cos(t1)}\Bigg), s =$$

$$-(2\, RootOf((4 \sin(t1)^2 l^2 - 24 \sin(t1) l - 16 \sin(t1)^2 - 48 \cos(t1) + 88) \_Z^2$$
$$+ (-4 \sin(t1)^2 l + 8 \sin(t1)^2 + 12 \cos(t1) + 12 \sin(t1) - 8) \_Z$$
$$+ 4 \sin(t1)^4 l^2 - 16 \sin(t1)^4 l - 24 \cos(t1) l \sin(t1)^2 - 4 \sin(t1)^2 l^2$$
$$- 24 \sin(t1)^3 l + 16 \sin(t1)^4 + 48 \cos(t1) \sin(t1)^2 + 16 \sin(t1)^2 l$$
$$+ 48 \sin(t1)^3 + 72 \cos(t1) \sin(t1) + 24 \sin(t1) l - 16 \sin(t1)^2 - 48 \sin(t1)$$
$$- 35) \sin(t1)^3 l + 2\, RootOf((4 \sin(t1)^2 l^2 - 24 \sin(t1) l - 16 \sin(t1)^2$$
$$- 48 \cos(t1) + 88) \_Z^2 + (-4 \sin(t1)^2 l + 8 \sin(t1)^2 + 12 \cos(t1)$$
$$+ 12 \sin(t1) - 8) \_Z + 4 \sin(t1)^4 l^2 - 16 \sin(t1)^4 l - 24 \cos(t1) l \sin(t1)^2$$
$$- 4 \sin(t1)^2 l^2 - 24 \sin(t1)^3 l + 16 \sin(t1)^4 + 48 \cos(t1) \sin(t1)^2$$
$$+ 16 \sin(t1)^2 l + 48 \sin(t1)^3 + 72 \cos(t1) \sin(t1) + 24 \sin(t1) l$$
$$- 16 \sin(t1)^2 - 48 \sin(t1) - 35) \sin(t1) \cos(t1)^2 l + 4 \sin(t1)^2 \cos(t1) l^2$$
$$- 4\, RootOf((4 \sin(t1)^2 l^2 - 24 \sin(t1) l - 16 \sin(t1)^2 - 48 \cos(t1)$$
$$+ 88) \_Z^2 + (-4 \sin(t1)^2 l + 8 \sin(t1)^2 + 12 \cos(t1) + 12 \sin(t1) - 8) \_Z$$
$$+ 4 \sin(t1)^4 l^2 - 16 \sin(t1)^4 l - 24 \cos(t1) l \sin(t1)^2 - 4 \sin(t1)^2 l^2$$
$$- 24 \sin(t1)^3 l + 16 \sin(t1)^4 + 48 \cos(t1) \sin(t1)^2 + 16 \sin(t1)^2 l$$
$$+ 48 \sin(t1)^3 + 72 \cos(t1) \sin(t1) + 24 \sin(t1) l - 16 \sin(t1)^2 - 48 \sin(t1)$$
$$- 35) \sin(t1)^3 - 4\, RootOf((4 \sin(t1)^2 l^2 - 24 \sin(t1) l - 16 \sin(t1)^2$$
$$- 48 \cos(t1) + 88) \_Z^2 + (-4 \sin(t1)^2 l + 8 \sin(t1)^2 + 12 \cos(t1)$$
$$+ 12 \sin(t1) - 8) \_Z + 4 \sin(t1)^4 l^2 - 16 \sin(t1)^4 l - 24 \cos(t1) l \sin(t1)^2$$
$$- 4 \sin(t1)^2 l^2 - 24 \sin(t1)^3 l + 16 \sin(t1)^4 + 48 \cos(t1) \sin(t1)^2$$
$$+ 16 \sin(t1)^2 l + 48 \sin(t1)^3 + 72 \cos(t1) \sin(t1) + 24 \sin(t1) l$$
$$- 16 \sin(t1)^2 - 48 \sin(t1) - 35) \sin(t1) \cos(t1)^2 - 8 \cos(t1) l \sin(t1)^2$$
$$- 12\, RootOf((4 \sin(t1)^2 l^2 - 24 \sin(t1) l - 16 \sin(t1)^2 - 48 \cos(t1)$$
$$+ 88) \_Z^2 + (-4 \sin(t1)^2 l + 8 \sin(t1)^2 + 12 \cos(t1) + 12 \sin(t1) - 8) \_Z$$
$$+ 4 \sin(t1)^4 l^2 - 16 \sin(t1)^4 l - 24 \cos(t1) l \sin(t1)^2 - 4 \sin(t1)^2 l^2$$
$$- 24 \sin(t1)^3 l + 16 \sin(t1)^4 + 48 \cos(t1) \sin(t1)^2 + 16 \sin(t1)^2 l$$

$$+ 48 \sin(t1)^3 + 72 \cos(t1) \sin(t1) + 24 \sin(t1)\, l - 16 \sin(t1)^2 - 48 \sin(t1)$$
$$- 35) \sin(t1)^2 - 12\, RootOf((4 \sin(t1)^2 l^2 - 24 \sin(t1)\, l - 16 \sin(t1)^2$$
$$- 48 \cos(t1) + 88)\, \_Z^2 + (-4 \sin(t1)^2 l + 8 \sin(t1)^2 + 12 \cos(t1)$$
$$+ 12 \sin(t1) - 8)\, \_Z + 4 \sin(t1)^4 l^2 - 16 \sin(t1)^4 l - 24 \cos(t1)\, l \sin(t1)^2$$
$$- 4 \sin(t1)^2 l^2 - 24 \sin(t1)^3 l + 16 \sin(t1)^4 + 48 \cos(t1) \sin(t1)^2$$
$$+ 16 \sin(t1)^2 l + 48 \sin(t1)^3 + 72 \cos(t1) \sin(t1) + 24 \sin(t1)\, l$$
$$- 16 \sin(t1)^2 - 48 \sin(t1) - 35) \cos(t1)^2 + 12 \sin(t1)^2 l$$
$$- 24 \sin(t1) \cos(t1)\, l + 4\, RootOf((4 \sin(t1)^2 l^2 - 24 \sin(t1)\, l - 16 \sin(t1)^2$$
$$- 48 \cos(t1) + 88)\, \_Z^2 + (-4 \sin(t1)^2 l + 8 \sin(t1)^2 + 12 \cos(t1)$$
$$+ 12 \sin(t1) - 8)\, \_Z + 4 \sin(t1)^4 l^2 - 16 \sin(t1)^4 l - 24 \cos(t1)\, l \sin(t1)^2$$
$$- 4 \sin(t1)^2 l^2 - 24 \sin(t1)^3 l + 16 \sin(t1)^4 + 48 \cos(t1) \sin(t1)^2$$
$$+ 16 \sin(t1)^2 l + 48 \sin(t1)^3 + 72 \cos(t1) \sin(t1) + 24 \sin(t1)\, l$$
$$- 16 \sin(t1)^2 - 48 \sin(t1) - 35) \sin(t1) + 4\, RootOf((4 \sin(t1)^2 l^2$$
$$- 24 \sin(t1)\, l - 16 \sin(t1)^2 - 48 \cos(t1) + 88)\, \_Z^2 + (-4 \sin(t1)^2 l$$
$$+ 8 \sin(t1)^2 + 12 \cos(t1) + 12 \sin(t1) - 8)\, \_Z + 4 \sin(t1)^4 l^2 - 16 \sin(t1)^4 l$$
$$- 24 \cos(t1)\, l \sin(t1)^2 - 4 \sin(t1)^2 l^2 - 24 \sin(t1)^3 l + 16 \sin(t1)^4$$
$$+ 48 \cos(t1) \sin(t1)^2 + 16 \sin(t1)^2 l + 48 \sin(t1)^3 + 72 \cos(t1) \sin(t1)$$
$$+ 24 \sin(t1)\, l - 16 \sin(t1)^2 - 48 \sin(t1) - 35) \cos(t1) + 24 \cos(t1) \sin(t1)$$
$$- 37 \sin(t1) + 35 \cos(t1)) \Big/ (2\, (2\, RootOf((4 \sin(t1)^2 l^2 - 24 \sin(t1)\, l$$
$$- 16 \sin(t1)^2 - 48 \cos(t1) + 88)\, \_Z^2 + (-4 \sin(t1)^2 l + 8 \sin(t1)^2$$
$$+ 12 \cos(t1) + 12 \sin(t1) - 8)\, \_Z + 4 \sin(t1)^4 l^2 - 16 \sin(t1)^4 l$$
$$- 24 \cos(t1)\, l \sin(t1)^2 - 4 \sin(t1)^2 l^2 - 24 \sin(t1)^3 l + 16 \sin(t1)^4$$
$$+ 48 \cos(t1) \sin(t1)^2 + 16 \sin(t1)^2 l + 48 \sin(t1)^3 + 72 \cos(t1) \sin(t1)$$
$$+ 24 \sin(t1)\, l - 16 \sin(t1)^2 - 48 \sin(t1) - 35) \sin(t1)^3 l$$
$$+ 2\, RootOf((4 \sin(t1)^2 l^2 - 24 \sin(t1)\, l - 16 \sin(t1)^2 - 48 \cos(t1)$$
$$+ 88)\, \_Z^2 + (-4 \sin(t1)^2 l + 8 \sin(t1)^2 + 12 \cos(t1) + 12 \sin(t1) - 8)\, \_Z$$
$$+ 4 \sin(t1)^4 l^2 - 16 \sin(t1)^4 l - 24 \cos(t1)\, l \sin(t1)^2 - 4 \sin(t1)^2 l^2$$
$$- 24 \sin(t1)^3 l + 16 \sin(t1)^4 + 48 \cos(t1) \sin(t1)^2 + 16 \sin(t1)^2 l$$
$$+ 48 \sin(t1)^3 + 72 \cos(t1) \sin(t1) + 24 \sin(t1)\, l - 16 \sin(t1)^2 - 48 \sin(t1)$$
$$- 35) \sin(t1) \cos(t1)^2 l - 4\, RootOf((4 \sin(t1)^2 l^2 - 24 \sin(t1)\, l$$
$$- 16 \sin(t1)^2 - 48 \cos(t1) + 88)\, \_Z^2 + (-4 \sin(t1)^2 l + 8 \sin(t1)^2$$
$$+ 12 \cos(t1) + 12 \sin(t1) - 8)\, \_Z + 4 \sin(t1)^4 l^2 - 16 \sin(t1)^4 l$$
$$- 24 \cos(t1)\, l \sin(t1)^2 - 4 \sin(t1)^2 l^2 - 24 \sin(t1)^3 l + 16 \sin(t1)^4$$
$$+ 48 \cos(t1) \sin(t1)^2 + 16 \sin(t1)^2 l + 48 \sin(t1)^3 + 72 \cos(t1) \sin(t1)$$

$+ 24 \sin(t1)\, l - 16 \sin(t1)^2 - 48 \sin(t1) - 35)\, \sin(t1)^3$
$- 4\, RootOf((4 \sin(t1)^2\, l^2 - 24 \sin(t1)\, l - 16 \sin(t1)^2 - 48 \cos(t1)$
$+ 88)\, \_Z^2 + (-4 \sin(t1)^2\, l + 8 \sin(t1)^2 + 12 \cos(t1) + 12 \sin(t1) - 8)\, \_Z$
$+ 4 \sin(t1)^4\, l^2 - 16 \sin(t1)^4\, l - 24 \cos(t1)\, l \sin(t1)^2 - 4 \sin(t1)^2\, l^2$
$- 24 \sin(t1)^3\, l + 16 \sin(t1)^4 + 48 \cos(t1) \sin(t1)^2 + 16 \sin(t1)^2\, l$
$+ 48 \sin(t1)^3 + 72 \cos(t1) \sin(t1) + 24 \sin(t1)\, l - 16 \sin(t1)^2 - 48 \sin(t1)$
$- 35)\, \sin(t1) \cos(t1)^2 - 6\, RootOf((4 \sin(t1)^2\, l^2 - 24 \sin(t1)\, l$
$- 16 \sin(t1)^2 - 48 \cos(t1) + 88)\, \_Z^2 + (-4 \sin(t1)^2\, l + 8 \sin(t1)^2$
$+ 12 \cos(t1) + 12 \sin(t1) - 8)\, \_Z + 4 \sin(t1)^4\, l^2 - 16 \sin(t1)^4\, l$
$- 24 \cos(t1)\, l \sin(t1)^2 - 4 \sin(t1)^2\, l^2 - 24 \sin(t1)^3\, l + 16 \sin(t1)^4$
$+ 48 \cos(t1) \sin(t1)^2 + 16 \sin(t1)^2\, l + 48 \sin(t1)^3 + 72 \cos(t1) \sin(t1)$
$+ 24 \sin(t1)\, l - 16 \sin(t1)^2 - 48 \sin(t1) - 35)\, \sin(t1)^2$
$- 6\, RootOf((4 \sin(t1)^2\, l^2 - 24 \sin(t1)\, l - 16 \sin(t1)^2 - 48 \cos(t1)$
$+ 88)\, \_Z^2 + (-4 \sin(t1)^2\, l + 8 \sin(t1)^2 + 12 \cos(t1) + 12 \sin(t1) - 8)\, \_Z$
$+ 4 \sin(t1)^4\, l^2 - 16 \sin(t1)^4\, l - 24 \cos(t1)\, l \sin(t1)^2 - 4 \sin(t1)^2\, l^2$
$- 24 \sin(t1)^3\, l + 16 \sin(t1)^4 + 48 \cos(t1) \sin(t1)^2 + 16 \sin(t1)^2\, l$
$+ 48 \sin(t1)^3 + 72 \cos(t1) \sin(t1) + 24 \sin(t1)\, l - 16 \sin(t1)^2 - 48 \sin(t1)$
$- 35)\, \cos(t1)^2 + 4\, RootOf((4 \sin(t1)^2\, l^2 - 24 \sin(t1)\, l - 16 \sin(t1)^2$
$- 48 \cos(t1) + 88)\, \_Z^2 + (-4 \sin(t1)^2\, l + 8 \sin(t1)^2 + 12 \cos(t1)$
$+ 12 \sin(t1) - 8)\, \_Z + 4 \sin(t1)^4\, l^2 - 16 \sin(t1)^4\, l - 24 \cos(t1)\, l \sin(t1)^2$
$- 4 \sin(t1)^2\, l^2 - 24 \sin(t1)^3\, l + 16 \sin(t1)^4 + 48 \cos(t1) \sin(t1)^2$
$+ 16 \sin(t1)^2\, l + 48 \sin(t1)^3 + 72 \cos(t1) \sin(t1) + 24 \sin(t1)\, l$
$- 16 \sin(t1)^2 - 48 \sin(t1) - 35)\, \sin(t1) - \sin(t1)))]]$

6) This example calculates invariant elements by a collineation using eigenvalues and eigenvectors. So we can determine the type of collineation and the

C := *Matrix*([[1., 0., 1.], [0., 2., 0.], [-3., 0., 5.]]);

$$C := \begin{bmatrix} 1. & 0. & 1. \\ 0. & 2. & 0. \\ -3. & 0. & 5. \end{bmatrix} \quad \textbf{(11)}$$

B := *MatrixInverse*(C);

$$B := \begin{bmatrix} 0.625000000000000 & 0. & -0.125000000000000 \\ -0. & 0.500000000000000 & 0. \\ 0.375000000000000 & 0. & 0.125000000000000 \end{bmatrix} \quad (12)$$

Tr := *Transpose*(*B*);

$$Tr := \begin{bmatrix} 0.625000000000000 & -0. & 0.375000000000000 \\ 0. & 0.500000000000000 & 0. \\ -0.125000000000000 & 0. & 0.125000000000000 \end{bmatrix} \quad (13)$$

Eigenvalues(*C*);
Eigenvalues(*B*);

$$\begin{bmatrix} 2. + 0.\,\mathrm{I} \\ 4. + 0.\,\mathrm{I} \\ 2. + 0.\,\mathrm{I} \end{bmatrix}$$

$$\begin{bmatrix} 0.500000000000000 + 0.\,\mathrm{I} \\ 0.250000000000000 + 0.\,\mathrm{I} \\ 0.500000000000000 + 0.\,\mathrm{I} \end{bmatrix} \quad (14)$$

Eigenvalues(*Tr*);

$$\begin{bmatrix} 0.500000000000000 + 0.\,\mathrm{I} \\ 0.250000000000000 + 0.\,\mathrm{I} \\ 0.500000000000000 + 0.\,\mathrm{I} \end{bmatrix} \quad (15)$$

Eigenvectors(*C*);
Eigenvectors(*B*);

$$\begin{bmatrix} 2. + 0.\,\mathrm{I} \\ 4. + 0.\,\mathrm{I} \\ 2. + 0.\,\mathrm{I} \end{bmatrix},$$

$$\begin{bmatrix} -0.707106781186547 + 0.\,\mathrm{I} & -0.316227766016838 + 0.\,\mathrm{I} & 0.\,\mathrm{I} \\ 0.\,\mathrm{I} & 0.\,\mathrm{I} & 1. + 0.\,\mathrm{I} \\ -0.707106781186547 + 0.\,\mathrm{I} & -0.948683298050514 + 0.\,\mathrm{I} & 0.\,\mathrm{I} \end{bmatrix}$$

$$\begin{bmatrix} 0.500000000000000 + 0.\,\mathrm{I} \\ 0.250000000000000 + 0.\,\mathrm{I} \\ 0.500000000000000 + 0.\,\mathrm{I} \end{bmatrix}, \quad (16)$$

$$\begin{bmatrix} 0.707106781186547 + 0.\,I & -0.316227766016838 + 0.\,I & 0.\,I \\ 0.\,I & 0.\,I & -1. + 0.\,I \\ 0.707106781186547 + 0.\,I & -0.948683298050514 + 0.\,I & 0.\,I \end{bmatrix}$$

Eigenvectors(*Tr*);

$$\begin{bmatrix} 0.500000000000000 + 0.\,I \\ 0.250000000000000 + 0.\,I \\ 0.500000000000000 + 0.\,I \end{bmatrix}, \quad \textbf{(17)}$$

$$\begin{bmatrix} 0.948683298050514 + 0.\,I & -0.707106781186547 + 0.\,I & 0.\,I \\ 0.\,I & 0.\,I & 1. + 0.\,I \\ -0.316227766016838 + 0.\,I & 0.707106781186547 + 0.\,I & 0.\,I \end{bmatrix}$$

Observe that Y(0,1,0) and P(1,0,1) (eigenvectors to 2) belong to the line l[−3,0,1], eigenvector to1/4. **l is the axis.**
Observe again that the lines m[−1,0,1] and n[0,1,0] (eigenvectors to 1/2) go through C(1/3,0,1), eigenvector to4, **C is the centre.**

7) We study also an isometry in 3D

C := *Matrix*$\left(\left[\left[\frac{1}{3}, \frac{2}{3}, \frac{2}{3}, -\frac{2}{3}\right], \left[-\frac{2}{3}, \frac{2}{3}, -\frac{1}{3}, \frac{4}{3}\right], \left[\frac{2}{3}, \frac{1}{3}, -\frac{2}{3}, \frac{2}{3}\right], [0, 0, 0, 1]\right]\right)$;
Eigenvalues(*C*);
Eigenvectors(*C*);

$$C := \begin{bmatrix} \frac{1}{3} & \frac{2}{3} & \frac{2}{3} & -\frac{2}{3} \\ -\frac{2}{3} & \frac{2}{3} & -\frac{1}{3} & \frac{4}{3} \\ \frac{2}{3} & \frac{1}{3} & -\frac{2}{3} & \frac{2}{3} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 \\ -1 \\ \frac{2}{3} - \frac{I\sqrt{5}}{3} \\ \frac{2}{3} + \frac{I\sqrt{5}}{3} \end{bmatrix}$$

$$\begin{bmatrix} 1 \\ -1 \\ \frac{2}{3} + \frac{I\sqrt{5}}{3} \\ \frac{2}{3} - \frac{I\sqrt{5}}{3} \end{bmatrix}, \tag{18}$$

$$\begin{bmatrix} 1 & -\frac{1}{2} & \frac{6\left(-\frac{1}{3} + \frac{I\sqrt{5}}{3}\right)}{\left(\frac{2}{3} + \frac{I\sqrt{5}}{3}\right)\left(1 + I\sqrt{5}\right)} & \frac{6\left(-\frac{1}{3} - \frac{I\sqrt{5}}{3}\right)}{\left(\frac{2}{3} - \frac{I\sqrt{5}}{3}\right)\left(1 - I\sqrt{5}\right)} \\ 1 & 0 & \frac{-\frac{5}{3} + \frac{2I\sqrt{5}}{3}}{\frac{2}{3} + \frac{I\sqrt{5}}{3}} & \frac{-\frac{5}{3} - \frac{2I\sqrt{5}}{3}}{\frac{2}{3} - \frac{I\sqrt{5}}{3}} \\ 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

Observe that the point C(1,1,1) is invariant and the vector n=(−1/2, 0, 1) is the normal vector to an invariant planes, not pointwise invariant, with equation x −2z+1=0.
Observe that the isometry is a composition of the reflection in the plane x−2z+1=0 och the rotation with axis th enormal to the plane x−2z+1 = 0 through C and rotation angle the angle with cosine 2/3 and sine \sqrt{5}/3

8) The following example is an elation

C := *Matrix*([[1., 1., 0.], [0., 1., 0.], [0., 0., 1.]]);
Eigenvalues(*C*);

$$\begin{bmatrix} 1. & 1. & 0. \\ 0. & 1. & 0. \\ 0. & 0. & 1. \end{bmatrix} \quad \textbf{(19)}$$

Eigenvectors(*C*);

$$\begin{bmatrix} 1.+0.\,I \\ 1.+0.\,I \\ 1.+0.\,I \end{bmatrix}, \begin{bmatrix} 1.+0.\,I & -1.+0.\,I & 0.+0.\,I \\ 0.+0.\,I & 2.22044604925031\;10^{-16}+0.\,I & 0.+0.\,I \\ 0.+0.\,I & -0.+0.\,I & 1.+0.\,I \end{bmatrix} \quad \textbf{(20)}$$

F := *Matrix*([[1., 0., 0.], [-1., 1., 0.], [0., 0., 1.]]);
Eigenvalues(*F*);

$$\begin{bmatrix} 1. & 0. & 0. \\ -1. & 1. & 0. \\ 0. & 0. & 1. \end{bmatrix} \quad \textbf{(21)}$$

Eigenvectors(*F*);

$$\begin{bmatrix} 1.+0.\,I \\ 1.+0.\,I \\ 1.+0.\,I \end{bmatrix}, \begin{bmatrix} 0.+0.\,I & -2.22044604925031\;10^{-16}+0.\,I & 0.+0.\,I \\ 1.+0.\,I & -1.+0.\,I & 0.+0.\,I \\ 0.+0.\,I & -0.+0.\,I & 1.+0.\,I \end{bmatrix} \quad \textbf{(22)}$$

The lines r[0,1,0] (y=0)and s[0,0,1] (z=0) intersect at P[1,0,0]. So P is the centre. The points P[1,0,0] and Q[0,0,1] belong to the line r[0,1,0] (y=0) so r is the axis. As the centre belongs to the axis we have a elation

9) Finally the following example is a collineation and its invariant elements. It has three points and three lines invariants, and it is NOT a perspectivistic collineation.

A := *Matrix*([[-1., -1., 2.], [-5., -1., 3.], [3., 0.,-1.]]);

$$A := \begin{bmatrix} -1. & -1. & 2. \\ -5. & -1. & 3. \\ 3. & 0. & -1. \end{bmatrix} \quad \textbf{(23)}$$

B := *Transpose*(*MatrixInverse*(*A*));

$$B := \begin{bmatrix} 1.00000000000000 & 4.00000000000001 & 3.00000000000000 \\ -1.00000000000000 & -5.00000000000001 & -3.00000000000000 \\ -1.00000000000000 & -7.00000000000001 & -4.00000000000000 \end{bmatrix} \quad \textbf{(24)}$$

Eigenvectors(*A*);

$Eigenvectors(B);$

$$\left[\begin{array}{c} -4.66792869559189+0.\,I \\ 1.78775903903729+0.\,I \\ -0.119830343445404+0.\,I \end{array}\right], [[-0.415280907253006+0.\,I, -0.624766541251697+0.\,I, 0.144545798199293+0.\,I], [-0.843903994776894+0.\,I, 0.397034562962171+0.\,I, 0.858124740559903+0.\,I], [0.339658380833921+0.\,I, -0.672332004850156+0.\,I, 0.492674783058692+0.\,I]]$$

$$\left[\begin{array}{c} -8.34513171912598+0.\,I \\ 0.559359498771433+0.\,I \\ -0.214227779645465+0.\,I \end{array}\right], [[0.454010394091017+0.\,I, -0.898570015650219+0.\,I, -0.775090647914976+0.\,I], [-0.515821456363572+0.\,I, 0.322327002824651+0.\,I, -0.211315625858943+0.\,I], [-0.726500369726185+0.\,I, -0.297787223071176+0.\,I, 0.595466366625845+0.\,I]] \quad (25)$$

$Linear(\{x+y+z-3, 2\,x-y+2\,z-3, x+y-2\,z\}, \{x, y, z\});$

$$\{x=1, y=1, z=1\} \quad (26)$$

$solve(\{x+y+z=3, 2\,x-y+2\,z=3, x+y-2\,z=0\}, [x, y, z]);$

$$[[x=1, y=1, z=1]] \quad (27)$$

$R := Matrix([[1, 1, 1], [2,-1, 2], [1, 1,-2]]);$
$v := Vector([3, 2, 0]);$
$LinearSolve(R, v);$

$$R := \left[\begin{array}{ccc} 1 & 1 & 1 \\ 2 & -1 & 2 \\ 1 & 1 & -2 \end{array}\right]$$

$$v := \left[\begin{array}{c} 3 \\ 2 \\ 0 \end{array}\right]$$

$$\begin{bmatrix} \frac{2}{3} \\ \frac{4}{3} \\ 1 \end{bmatrix} \tag{28}$$

$K := \langle\langle 1, 2, 1\rangle | \langle 1, -1, 1\rangle | \langle 1, 2, -2\rangle\rangle;$
$LinearSolve(K, v);$

$$K := \begin{bmatrix} 1 & 1 & 1 \\ 2 & -1 & 2 \\ 1 & 1 & -2 \end{bmatrix}$$

$$\begin{bmatrix} \frac{2}{3} \\ \frac{4}{3} \\ 1 \end{bmatrix} \tag{29}$$