Exercises on Stereographic Projection

13.8.1 Find the image of the circle $\{z ; |z|=r\}$ under stereographic projection

We can do it in two ~~ways~~ ways

1) $\mathcal{C}_r = \{z ; |z|=r\}$ has coordinates $z = r\cos\theta + i\sin\theta$

So $\varphi_N(z) = \left(\frac{2r\cos\theta}{r^2+1}, \frac{2r\sin\theta}{r^2+1}, \frac{r^2-1}{r^2+1}\right)$ parametrised by $\theta \in [0, 2\pi)$

We see that the ~~points~~ in $\varphi_N(\mathcal{C}_r)$ satisfy

$$x_3 = \frac{r^2-1}{r^2+1}$$

So $\varphi_N(\mathcal{C}_r)$ is the parallel of latitude λ_r s.t. $\sin\lambda = \frac{r^2-1}{r^2+1}$

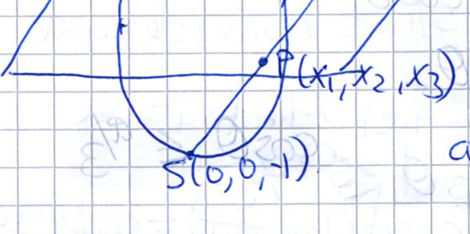
2) $\varphi_N(\mathcal{C}_r)$ is the intersection of S^2 with a plane $a_1x_1 + a_2x_2 + a_3x_3 = b$ where the equation $x^2+y^2=r^2$ of $\mathcal{C}_r$ is the equation $(a_3-b)(x^2+y^2) + 2a_1x + 2a_2y = a_3+b$

so $a_1 = a_2 = 0$ and $\begin{cases} a_3 - b = 1 \\ a_3 + b = r^2 \end{cases}$ $\quad a_3 = 1+r^2$, $b = r^2-1$

i.e. $\quad x_3 = \frac{r^2-1}{r^2+1}$

13.8.~~4~~4 Show that projection $\varphi_S : S^2 \to \mathbb{C}$ from the "south pole" $S(0,0,-1)$ is given by

$\varphi_S(x_1, x_2, x_3) = \frac{x_1}{1+x_3} + i\frac{x_2}{1+x_3}$



We have to consider the lines $\begin{cases} X = \lambda x_1 \\ Y = \lambda x_2 \\ Z = -1 + \lambda(x_3+1) \end{cases}$

For the points $Z=0$ $\lambda = \frac{1}{x_3+1}$

and $x = \frac{x_1}{1+x_3}$, $y = \frac{x_2}{1+x_3}$

13.9.2 Show that if $P = \varphi_N(z)$ and $Q = \varphi_N(w)$, $z = x+iy$, $w = r+is$, are diametrically opposite iff $w = -\frac{1}{\bar{z}}$

$P = \varphi_N(z)$ diametrically opposite $\iff \varphi_N(w) = -\varphi_N(z)$
$Q = \varphi_N(w)$

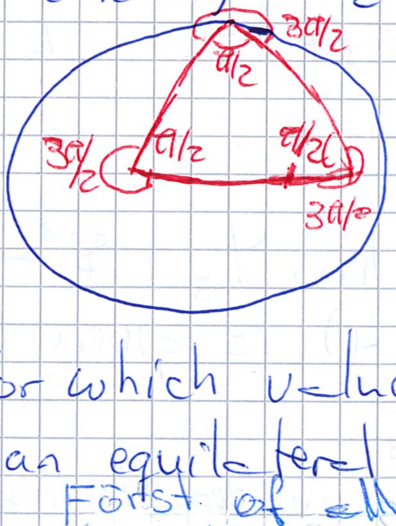
$$\iff \left(\frac{2r}{r^2+s^2+1}, \frac{2s}{r^2+s^2+1}, \frac{r^2+s^2-1}{r^2+s^2+1}\right) = \left(\frac{-2x}{x^2+y^2+1}, \frac{-2y}{x^2+y^2+1}, \frac{x^2+y^2-1}{x^2+y^2+1}\right)$$

$$\begin{cases} \dfrac{r}{(r^2+s^2+1)} = \dfrac{-x}{(x^2+y^2+1)} \\ \dfrac{s}{(r^2+s^2+1)} = \dfrac{-y}{(x^2+y^2+1)} \\ \dfrac{r^2+s^2-1}{r^2+s^2+1} = \dfrac{x^2+y^2-1}{x^2+y^2+1} \end{cases} \Longrightarrow \begin{matrix} r = \dfrac{-x}{x^2+y^2} \\ y = \dfrac{-y}{x^2+y^2} \end{matrix} \quad w = \frac{-z}{|z|^2} = -\frac{1}{\bar{z}}$$

Exercises on Spherical Trigonometry

5.3.1 a) Calculate the area of a spherical triangle all of whose angles are $\pi/2$ $\quad \text{Area}(T_{\pi/2\,\pi/2\,\pi/2}) = \underline{\pi/2} = \frac{1}{8}4\pi$

$\text{Area}(T_{3\pi/2\,3\pi/2\,3\pi/2}) = \frac{7\pi}{2} = 4\pi - \frac{4\pi}{8}$



The two triangles covering the sphere

See last Exercise!!

5.3.2 For which values of θ is it possible to construct an equilateral spherical triangle $\neq d$

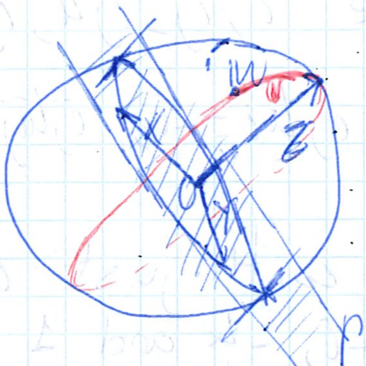
First of all by sine-rule $\delta(A,B) = \delta(A,C) = \delta(B,C)$ since their sines coincide and $0 < \delta(A,B) < \pi$.

Secondly as the only parallel that is a great circle is the Equator $\Longrightarrow d \leq \pi/2$

Cosine-rule $\cos d = \cos^2 d + \sin^2 d \cos\theta$ $\begin{cases} \cos d\,(1-\cancel{\cos d}) = \cos\theta\,(1-\cos^2 d) \\ 2\cos^2 d/2 - 1 = (1-2\sin^2\theta/2)\,2\cos^2 d/2 \end{cases}$

$1 - 2\sin^2 d/2 = \boxed{1 = 4\cos^2 d/2 \sin^2\theta/2} \iff \sin\theta/2 = \frac{1}{2\cos d/2}$

Exercise 5.1.2



The normal to the plane through x and y has the direction $x \times y$.
It intersects the sphere at a point of distance 1 from 0:

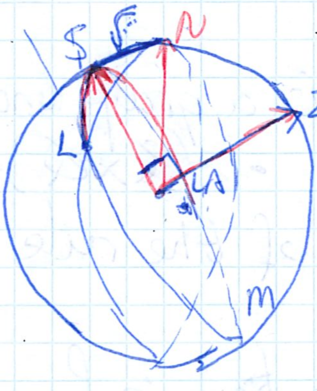
$$z = \frac{x \times y}{|x \times y|}$$

If w lies on the same side of the plane as z then $\delta(w, z) = \arccos(w \cdot z)$

$$\cos \delta(w, z) = w \cdot z = \underbrace{w \cdot (x \times y) = [w, x, y]}_{\text{the complet product}}$$

Exercise 5.1-3

Suppose that an airplane flies on the shortest rout from London ($\alpha_L = 54°$, $\beta_L = 0°$) to LA ($\alpha_{LA} = 34°$, $\beta_{LA} = -151°$) How close the aircraft get to the north pole. x_L = coordinates to London, y_{LA} = coordinates to LA.

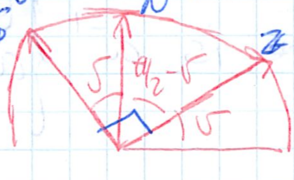


$$z = \frac{x_L \times y_{LA}}{|x_L \times y_{LA}|}$$ normal to plane of x_L and y_{LA}, and so m

S = point on the great circle m nearest $N \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

Observe that $S \perp z$ and | with other words

Now $\cos\delta(S, N)$ = latitude of z |



$\cos(N, z) = \underline{\sin \delta(S, N)}$ $\quad$ $\cos \delta(S, N) = \underline{\sin \angle(N, z)}$

$\underline{N \cdot z}$ $\quad$ $= \underline{\sin}$

Exercises on Quaternions

6.2.1 Simplify the sequence $i, ij, ijk, ijki, \ldots$

As $i = i$, $ij = k$, $ijk = -1$, $ijki = -i$, $ijkij = -k$

and $ijkijk = 1$

we have a ~~cyclic~~ periodic sequence with just the six quaternions $i, k, -1, -i, -k$ and 1

6.2.2 Let $q = (a, x)$. Express a and x in terms ~~and~~ of q and $\bar{q}$

We have $a = \frac{q+\bar{q}}{2}$ and $x = \frac{q-\bar{q}}{2}$

6.2.3 Let q be a pure quaternion. Compute q^2 and show that $q^{-1} = \frac{-q}{|q|^2}$

$q = (0, x)$ and $q^2 = (0 - |x|^2, 0) = (-|q|^2, 0)$

So $q\left(\frac{-q}{|q|^2}\right) = -\frac{q^2}{|q|^2} = -\frac{(-|q|^2)}{|q|^2} = 1$.

i.e $q^{-1} = -\frac{q}{|q|^2}$

6.2.4 $Q = \{1, -1, i, -i, j, -j, k, -k\}$ is a group under multiplication

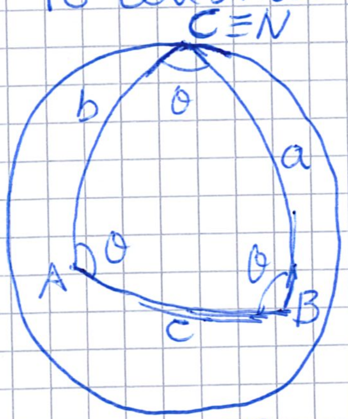
i) First we must controll that $\cdot : (Q \times Q) \to Q$ closed i.e $a \cdot b \in Q$ this follows of the rule of multiplication for $\mathbb{H}$.

ii) It is associative as multiplication is for the whole $\mathbb{H}$

iii) There is identity element $1_d = 1$ ($1 \cdot a = a$ $\forall a \in Q$)

iv) Given $a \in Q$, there is a^{-1} $\begin{cases} (-1)^{-1} = -1 \text{ (one)} \\ a = i, j, k, -i, -j, -k \\ \underline{a^{-1} = -a} \end{cases}$

5.3.2 For wich values of θ it is possible to construct an equilateral spherical triangle?



First of all $0 < a, b, c < \pi$

By sine rule $\dfrac{\sin\theta}{\sin a} = \dfrac{\sin\theta}{\sin b} = \dfrac{\sin\theta}{\sin c}$

We have that $a = b = c$ or

$a = b$ and $c = \pi - a$

By cosine rule $\cos c = \cos^2 a + \sin^2 a \cos\theta$

If $c = \pi - a$ we obtain $-\cos a = \cos^2 a + \sin^2 a \cos\theta$

If $c = a$ we have $\cos a = \cos^2 a + \sin^2 a \cos\theta$

In both cases as $\cos a - 1 \neq 0$ $(0 < a)$

and having $\cos a = 2\cos^2\frac{a}{2} - 1$, $1 + \cos a = 2\cos^2\frac{a}{2}$

and $\cos\theta = 1 - 2\sin^2\theta/2$, a and θ satisfy the equation $\sin\frac{\theta}{2} = \dfrac{1}{2\cos\frac{a}{2}}$ $\quad \cos\frac{a}{2} \neq 0$, since $a < \pi$

Notice that $\sin\theta/2$ is an increasing function of $\frac{a}{2}$. The minimum is obtain for $\cos\frac{a}{2} = 1$

Then $\sin\theta/2 = \frac{1}{2}$ $\quad \theta/2 = \pi/6$ and $\boxed{\theta = \pi/3}$

Observe that the maximum is obtained by the second triangle in Exercise 5.3.1.