Assignment I

Honours Linear Algebra TATA53 Spring 2024 Jonathan Nilsson

Instructions: Write your name, personal number, and program (if applicable) on the first page. The assignment should be solved individually. Your solutions should not depend on digital tools (unless otherwise stated). Write complete solutions with clear answers. You are allowed to discuss the problems with other students, but you should write your solutions individually, don't write anything that you don't understand yourself. Write your solutions on paper by hand and staple them together. Some problems marked ⚔ are harder, it is not necessary to solve all the problems. You should be prepared to present your solutions on *one* of the two associated seminars.
Deadline: 08.00AM on February 5 2024. Hand in the assignments in the compartment labelled TATA53 one floor up in the B-building just above entrance 21, or bring them to the *first* seminar starting 15 minutes later.

1. Find the reduced row echelon form of the matrix A below. Use your result to solve the linear system $AX = 0$.

$$A = \begin{pmatrix} 1 & i & 2 & 7 \\ 1-i & 1+i & 1 & 4-i \\ 2i & -2 & 1+2i & 3+8i \end{pmatrix}$$

2. Consider $V = \mathbb{R}^2$ as just a set. V becomes a real vector space under the following definitions of addition and multiplication by real numbers:

$$(x, y) \mathbin{\pmb{+}} (x', y') := (x + x' - 3, y + y' - 5)$$

$$\lambda \bullet (x, y) := (\lambda x - 3\lambda + 3, \lambda y - 5\lambda + 5)$$

   (a) What is the zero-vector (the additive identity) of the vector space V?
   (b) Find $-(2, 1)$, the additive inverse of the vector $(2, 1)$.
   (c) Verify that the vector space axiom $\lambda \bullet (\mu \bullet v) = (\lambda\mu) \bullet v$ holds in V.
   (d) Let $e_1 = (1, 1)$ and $e_2 = (2, 3)$. Is (e_1, e_2) a basis for V?

3. Find LU- and LDU-factorizations of

$$A = \begin{pmatrix} 1 & 1 & 2 & -1 \\ 1 & 2 & 3 & 1 \\ -1 & 2 & 4 & 2 \end{pmatrix}.$$

4. Let $F : \mathbb{R}^3 \to \mathbb{R}^3$ be given by $F(v) = (1, 2, 2) \times v$, the usual vector product on $\mathbb{R}^3$.

   (a) Show that F is linear, and find the matrix of F with respect to the standard basis in $\mathbb{R}^3$.
   (b) Find all eigenvalues and eigenvectors of F. Is F diagonalizable?
   (c) The same matrix can be viewed as a map $G : \mathbb{C}^3 \to \mathbb{C}^3$ (the complexification of F). Find all eigenvalues and eigenvectors of G. Is G diagonalizable?

5. Find the Cholesky-factorization of

$$B = \begin{pmatrix} 1 & 2 & 1 \\ 2 & 8 & 2-4i \\ 1 & 2+4i & 7 \end{pmatrix}.$$

6. Determine whether each statement below is true or false. Give a very short proof or a counterexample to each statement.

   (a) The determinant function $\det : \mathrm{Mat}_2(\mathbb{R}) \to \mathbb{R}$ is a linear map.

   (b) The set of 2×2 Hermitian matrices is a subspace of $\mathrm{Mat}_2(\mathbb{C})$.

   (c) In $\mathbb{R}^3/U$ we have $(1,2,3) + U = (3,2,1) + U$, where $U = \mathrm{Span}(1,0,-1)$.

   (d) $\mathrm{Mat}_n(\mathbb{R}) = \mathfrak{t} \oplus \mathfrak{s}$, where $\mathfrak{t}$ is the set of strictly upper triangular matrices, and $\mathfrak{s}$ is the set of symmetric matrices.

   (e) Commutativity in $\mathrm{Mat}_2(\mathbb{R})$ is transitive: If A commutes with B, and B commutes with C, then A commutes with C.

   (f) If $\dim V < \infty$ and $F : V \to V$ is linear, then $V = \ker(F) \oplus \mathrm{Im}(F)$.

7. Let $\mathcal{S}_n$ be the subset of $\mathrm{Mat}_n(\mathbb{R})$ consisting of matrices where the sum in each row and in each column are all equal. For example, below we have $A \in \mathcal{S}_2$ and $B, C \in \mathcal{S}_3$:

$$A = \begin{array}{|c|c|}\hline 4 & 3 \\ \hline 3 & 4 \\ \hline \end{array} \qquad B = \begin{array}{|c|c|c|}\hline 8 & 1 & 6 \\ \hline 3 & 5 & 7 \\ \hline 4 & 9 & 2 \\ \hline \end{array} \qquad C = \begin{array}{|c|c|c|}\hline 1 & 1 & 1 \\ \hline 1 & 1 & 1 \\ \hline 1 & 1 & 1 \\ \hline \end{array}$$

   (a) Show that $\mathcal{S}_n$ is a *subspace* of $\mathrm{Mat}_n(\mathbb{R})$.

   (b) Find a basis for $\mathcal{S}_2$ and determine the coordinates for A in that basis.

   (c) Find a basis for $\mathcal{S}_3$ and determine the coordinates for B and C in that basis.

   (d) Find the dimension of $\mathcal{S}_n$ for each n.

8. There are six buttons in a row, each with a light that is either on or off. When you press a button, its light switches on↔off, but it also switches the light on the adjacent buttons. At a given time, the status of the lights can be identified with a vector in $(\mathbb{Z}_2)^6$, where 1 means on and 0 means off. Pressing a button then corresponds to adding a certain vector to the status-vector, for example, pressing the third button corresponds to adding $(0,1,1,1,0,0)$. To solve the puzzle all the lights should be turned on.

   (a) Suppose the light-status is currently $(0,1,1,1,1,1)$. What buttons should be pressed to solve the puzzle?

   (b) Show that the puzzle is solvable regardless of the starting position.

   (c) Now consider the same puzzle but with n lights in a row. For what values of n is every starting position solvable?