Assignment II

Honours Linear Algebra · TATA53 · Spring 2024 · Jonathan Nilsson

Instructions: Write your name, personal number, and program (if applicable) on the first page. The assignment should be solved individually. Your solutions should not depend on digital tools (unless otherwise stated). Write complete solutions with clear answers. You are allowed to discuss the problems with other students, but you should write your solutions individually, don't write anything that you don't understand yourself. Write your solutions on paper by hand and staple them together. Some problems marked ⚔ are harder, it is not necessary to solve all the problems. You should be prepared to present your solutions on *one* of the two associated seminars.
Deadline: 08.00AM on March 4 2024. Hand in the assignments in the compartment labelled "TATA53 assignment submissions" one floor up in the B-building just above entrance 21, or bring them to the *first* seminar starting 15 minutes later.

1. A matrix A has characteristic and minimal polynomials [5]

$$p_A(t) = (t-1)^6(t-3)^3(t-5)^7 \quad \text{and} \quad m_A(t) = (t-1)^2(t-3)^2(t-5)^4.$$

We also know that the geometric multiplicity is 4 for two of the eigenvalues. Find the Jordan form of A.

2. Find A^{10}, $\cos(A)$, and e^{iA} for $A = \begin{pmatrix} \frac{\pi}{6} & 1 & 0 & 0 & 0 \\ 0 & \frac{\pi}{6} & 1 & 0 & 0 \\ 0 & 0 & \frac{\pi}{6} & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$. [5]

3. We define the *Manhattan norm* of a vector $(x, y) \in \mathbb{R}^2$ as $\|(x,y)\|_{\text{Mh}} := |x| + |y|$. [5]

   (a) Verify that $\|\cdot\|_{\text{Mh}}$ is indeed a norm on $\mathbb{R}^2$.

   (b) Show that the Manhattan norm can not be derived from an inner product: there is no inner product $\langle\cdot,\cdot\rangle$ for which $\|v\|_{\text{Mh}} = \sqrt{\langle v, v\rangle}$ for all $v \in \mathbb{R}^2$.
   Hint: Show that the Manhattan norm does not satisfy the parallelogram law.

   (c) With respect to the Manhattan norm, find every point in $\mathbb{R}^2$ whose distance to the origin $(0,0)$ equals its distance to $(3,1)$, illustrate your answer.

4. Jordanize the matrix A below by finding a matrix S and a matrix J in Jordan form such that $A = SJS^{-1}$. [9]

$$A = \begin{pmatrix} 5 & -2 & 1 & -3 \\ 0 & 2 & 0 & 0 \\ 1 & -1 & 3 & -1 \\ 3 & -2 & 1 & -1 \end{pmatrix}$$

5. Use the methods of this course to solve the following boundary value problem: [7]

$$\begin{cases} x_1'(t) = 5x_1(t) - 4x_2(t) \\ x_2'(t) = 9x_1(t) - 7x_2(t) \end{cases} \quad \text{with} \quad x_1(0) = 1 = x_2(0).$$

6. Determine whether each statement below is true or false, give a very short proof or a counterexample to each statement. All matrices A and B mentioned below should be assumed to be square of same size $n \times n$ where $n \geq 2$. [6]

   (a) A and B have the same reduced row echelon form if and only if they have the same Jordan form.

   (b) In any inner product space, if u and v have the same length, then $u + v$ is orthogonal to $u - v$.

   (c) The matrix e^A is always invertible.

   (d) In $\mathbb{R}^2$, the Manhattan norm of a vector is always less or equal than the standard Euclidean norm of the vector.

   (e) Any two matrices in Jordan form commute.

   (f) For the operator F on $\mathrm{Mat}_n(\mathbb{R})$ defined by $F(A) = A + A^T$, the minimal polynomial is $m_F(t) = t^2 - 2t$.

7. Let $\mathcal{P}$ be the real vector space of polynomials with real coefficients. Define [7]

$$\langle p(x), q(x) \rangle := \int_0^\infty p(x)q(x)e^{-x}dx.$$

   (a) Show that $\langle \cdot, \cdot \rangle$ is an inner product on $\mathcal{P}$.

   (b) Find the angle between the polynomials 1 and x.

   (c) Derive an explicit formula for $\langle x^m, x^n \rangle$.

   (d) Does there exist a polynomial q such that $\langle p(x), q(x) \rangle = p(2)$ for all $p \in \mathcal{P}_4$?

8. (a) A linear map F on a seven-dimensional vector space is defined by $F(e_1) = 0$, $F(e_2) = F(e_3) = e_1$, $F(e_4) = F(e_5) = e_2$, $F(e_6) = F(e_7) = e_3$ as indicated in graph below. Find the Jordan form of F. [6]

$$\begin{tikzcd}
e_4 \arrow[dr, mapsto] & & & \\
& e_2 \arrow[ddr, mapsto] & & \\
e_5 \arrow[ur, mapsto] & & & \\
& & e_1 \arrow[r, mapsto] & 0 \\
e_6 \arrow[dr, mapsto] & & & \\
& e_3 \arrow[uur, mapsto] & & \\
e_7 \arrow[ur, mapsto] & & &
\end{tikzcd}$$

   (b) Generalize your result to a tournament-like graph with n layers: let F_n be the operator on a $(2^n - 1)$-dimensional vector space where $F_n(e_1) = 0$ and $F_n(e_k) = e_{\lfloor \frac{k}{2} \rfloor}$ for $k > 1$. Find the Jordan form of F_n.