Assignment III

Honours Linear Algebra TATA53 Spring 2024 Jonathan Nilsson

Instructions: Write your name, personal number, and program (if applicable) on the first page. The assignment should be solved individually. Your solutions should not depend on digital tools, except for problems marked by ⌨. Write complete solutions with clear answers. You are allowed to discuss the problems with other students, but you should write your solutions individually, don't write anything that you don't understand yourself. Write your solutions on paper by hand and staple them together. Some problems marked ⚔ are harder, it is not necessary to solve all the problems. You should be prepared to present your solutions on *one* of the two associated seminars.

Deadline: 08.00AM on April 18 2024. Hand in the assignments in the compartment labelled "TATA53 assignment submissions" one floor up in the B-building just above entrance 21, or bring them to the *first* seminar starting 15 minutes later.

1. Find a QR-factorization of the matrix A below, in other words, find a unitary matrix Q and an upper triangular matrix R such that $A = QR$. [6]

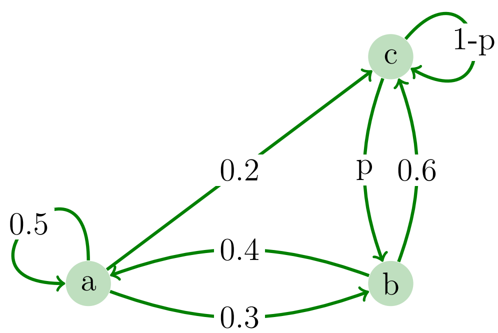
$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 3 & 4 \end{pmatrix}$$

2. Consider the real inner product space $\mathcal{C}[0,1]$ where the inner product is given by [6]

$$\langle f, g \rangle = \int_0^1 f(x)g(x)dx.$$

Find the polynomial $p \in \mathcal{P}_1$ that best approximates the function $f(x) = \sqrt{x}$ in the sense that $\|f - p\|$ is minimal. Sketch the graphs of p and of f in the same diagram.

3. A Markov process is described by the following graph where $0 \leq p \leq 1$ is a parameter. For each value of p, determine in what states we will spend the most and least time in the long run. [6]



4. Find the Moore-Penrose pseudo-inverse of the coefficient matrix A below and use it to find the least square solution to the linear system. [6]

$$\begin{cases} x + y &= 0 \\ x + iy &= 1 \\ ix - 2y &= 0 \\ x - y &= 1 \end{cases} \Leftrightarrow AX = b.$$

5. Let $A = \begin{pmatrix} 2 & \alpha & 1 \\ -\alpha & 1 & 0 \\ 1 & 0 & |\alpha| \end{pmatrix}$. For what values of $\alpha \in \mathbb{C}$ is the matrix $A \ldots$ [6]

(a) Irreducible? (b) Orthogonally diagonalizable? (c) Positive definite?

6. Determine whether each statement below is true or false. Motivate your answers well, for example by writing a short proof or by exhibiting a counter-example. [6]

(a) A square matrix $A \geq 0$ is irreducible if and only if A^2 is irreducible.

(b) $\sqrt{\begin{pmatrix} 5 & 4i \\ -4i & 5 \end{pmatrix}} = \begin{pmatrix} -2 & -i \\ i & -2 \end{pmatrix}$.

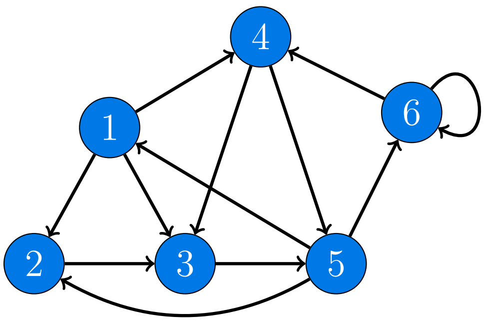
(c) If A has independent columns and if B is invertible, then $(AB)^+ = B^{-1}A^+$.

(d) If A is Hermitian and B is skew-Hermitian and A and B commute, then $A + B$ is normal.

(e) If $F : V \to V$ preserves the lengths of vectors: $\|F(v)\| = \|v\|$ for all $v \in V$, then F preserves *all* inner products: $\langle F(u), F(v)\rangle = \langle u, v\rangle$ for all $u, v \in V$.

(f) If $A > 0$ and $v = (v_1, \ldots, v_n)$ is an eigenvector of A, then $v_{\text{abs}} := (|v_1|, \ldots, |v_n|)$ is also an eigenvector of A.

7. Write a program that finds the PageRank of the graph below, use a dampening factor of $d = 0.9$. Answer with the ranking vector rounded to 2 decimals and normalized so that the sum of its entries is 1. Also determine if it is possible to add just one edge so that a different node becomes top-ranked. Either explain the important pieces of your code in words or print out your program and attach it. [7]



8. For each positive integer n, let G_n be the graph consisting of two loops of lengths 6 and n respectively with a single vertex in common. More precisely, G_n has $n + 5$ vertices $(v_1, \ldots, v_{n+5})$ and there are $n + 6$ edges: [7]

$$v_1 \to v_2 \to \cdots \to v_n \to v_1 \quad \text{and} \quad v_1 \to v_{n+1} \to v_{n+2} \to v_{n+3} \to v_{n+4} \to v_{n+5} \to v_1.$$

Let A_n be the adjacency-matrix of G_n. Determine for what values of n that A_n is primitive.